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- Each problem worked out in step-by-step detail, enabling you to understand the subject fully
- Will save you hours of time in finding solutions to problems

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THE TOPOLOGY PROBLEM SOLVER®

REGISTERED TRADEMARK

A Complete Solution Guide to Any Textbook

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THE TOPOLOGY PROBLEM SOLVER ®

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WHAT THIS BOOK IS FOR

Students have generally found topology a difficult subject to understand and learn. Despite the publication of hundreds of textbooks in this field, each one intended to provide an improvement over previous textbooks, students continue to remain perplexed as a result of the numerous conditions that must often be remembered and correlated in solving a problem. Various possible interpretations of terms used in topology have also contributed to much of the difficulties experienced by students.

In a study of the problem, REA found the following basic reasons underlying students' difficulties with topology taught in schools:

(a) No systematic rules of analysis have been developed which students may follow in a step-by-step manner to solve the usual problems encountered. This results from the fact that the numerous different conditions and principles which may be involved in a problem, lead to many possible different methods of solution. To prescribe a set of rules to be followed for each of the possible variations, would involve an enormous number of rules and steps to be searched through by students, and this task would perhaps be more burdensome than solving the problem directly with some accompanying trial and error to find the correct solution route.

(b) Textbooks currently available will usually explain a given principle in a few pages written by a professional who has an insight in the subject matter that is not shared by students. The explanations are often written in an abstract manner which leaves the students confused as to the application of the principle. The explanations given are not sufficiently detailed and extensive to make the student aware of the wide range of applications and different aspects of the principle being studied. The numerous possible variations of principles and their applications are usually not discussed, and it is left for the students to discover these for themselves while doing exercises. Accordingly, the average student is expected to rediscover that which has been long known and practiced, but not published or explained extensively.

(c) The examples usually following the explanation of a topic are too few in number and too simple to enable the student to obtain a thorough grasp of the principles involved. The explanations do not provide sufficient basis to enable a student to solve problems that may be subsequently assigned for homework or given on examinations.

The examples are presented in abbreviated form which leaves out much material between steps, and requires that students derive the omitted material

themselves. As a result, students find the examples difficult to understand—contrary to the purpose of the examples.

Examples are, furthermore, often worded in a confusing manner. They do not state the problem and then present the solution. Instead, they pass through a general discussion, never revealing what is to be solved for.

Examples, also, do not always include diagrams/graphs, wherever appropriate, and students do not obtain the training to draw diagrams or graphs to simplify and organize their thinking.

(d) Students can learn the subject only by doing the exercises themselves and reviewing them in class, to obtain experience in applying the principles with their different ramifications.

In doing the exercises by themselves, students find that they are required to devote considerably more time to topology than to other subjects of comparable credits, because they are uncertain with regard to the selection and application of the theorems and principles involved. It is also often necessary for students to discover those "tricks" not revealed in their texts (or review books), that make it possible to solve problems easily. Students must usually resort to methods of trial-and-error to discover these "tricks," and as a result they find that they may sometimes spend several hours to solve a single problem.

(e) When reviewing the exercises in classrooms, instructors usually request students to take turns in writing solutions on the boards and explaining them to the class. Students often find it difficult to explain in a manner that holds the interest of the class, and enables the remaining students to follow the material written on the boards. The remaining students seated in the class are, furthermore, too occupied with copying the material from the boards, to listen to the oral explanations and concentrate on the methods of solution.

This book is intended to aid students in topology overcoming the difficulties described, by supplying detailed illustrations of the solution methods which are usually not apparent to students. The solution methods are illustrated by problems selected from those that are most often assigned for class work and given on examinations. The problems are arranged in order of complexity to enable students to learn and understand a particular topic by reviewing the problems in sequence. The problems are illustrated with detailed step-by-step explanations, to save the students the large amount of time that is often needed to fill in the gaps that are usually found between steps of illustrations in textbooks or review/outline books.

The staff of REA considers topology a subject that is best learned by allowing students to view the methods of analysis and solution techniques themselves. This approach to learning the subject matter is similar to that practiced in various scientific laboratories, particularly in the medical fields.

In using this book, students may review and study the illustrated problems at their own pace; they are not limited to the time allowed for explaining problems on the board in class.

When students want to look up a particular type of problem and solution, they can readily locate it in the book by referring to the index which has been extensively prepared. It is also possible to locate a particular type of problem by glancing at just the material within the boxed portions. To facilitate rapid scanning of the problems, each problem has a heavy border around it. Furthermore, each problem is identified with a number immediately above the problem at the right-hand margin.

To obtain maximum benefit from the book, students should familiarize themselves with the section, "How To Use This Book," located in the front pages.

Special thanks are due to Dr. Nathan Busch for his technical editing of the book.

MAX FOGIEL, Ph.D.
Program Director

HOW TO USE THIS BOOK

This book can be an invaluable aid to students in topology as a supplement to their textbooks. The book is subdivided into 19 chapters, each dealing with a separate topic. The subject matter is developed beginning with the fundamental concepts of sentence calculus, sets, mappings, and extending through metric and topological spaces, continuity, homeomorphisms, axioms, compactness, connectedness, and homotopy theory.

TO LEARN AND UNDERSTAND A TOPIC THOROUGHLY

1. Refer to your class text and read the section pertaining to the topic. You should become acquainted with the principles discussed there. These principles, however, may not be clear to you at that time.

2. Then locate the topic you are looking for by referring to the "Table of Contents" in front of this book, "The Topology Problem Solver."

3. Turn to the page where the topic begins and review the problems under each topic, in the order given. For each topic, the problems are arranged in order of complexity, from the simplest to the more difficult. Some problems may appear similar to others, but each problem has been selected to illustrate a different point or solution method.

To learn and understand a topic thoroughly and retain its contents, it will be generally necessary for students to review the problems several times. Repeated review is essential in order to gain experience in recognizing the principles that should be applied, and in selecting the best solution technique.

TO FIND A PARTICULAR PROBLEM

To locate one or more problems related to a particular subject matter, refer to either the index, located at the back of the book or the indexes found at the beginning of each chapter. In using the indexes, be certain to note that the numbers given refer to problem numbers, not page numbers. This arrangement of the indexes is intended to facilitate finding a problem more rapidly, since two or more problems may appear on a page.

If a particular type of problem cannot be found readily, it is recommended that the student refer to the "Table of Contents" in the front pages,

and then turn to the chapter which is applicable to the problem being sought. By scanning or glancing at the material that is boxed, it will generally be possible to find problems related to the one being sought, without consuming considerable time. After the problems have been located, the solutions can be reviewed and studied in detail. For this purpose of locating problems rapidly, students should acquaint themselves with the organization of the book as found in the "Table of Contents."

In preparing for an exam, locate the topics to be covered on the exam in the "Table of Contents," and then review the problems under those topics several times. This should equip the student with what might be needed for the exam.

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CONTENTS

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CHAPTER 1

INTRODUCTION TO SENTENCE CALCULUS

Let α, β, γ, ... denote sentences, each of which has one of two logical values 0 or 1. We assign the value 0 to a false sentence and 1 to a true sentence. To express the fact that α is a true sentence we write

$$\alpha \equiv 1 \text{ or } \beta \equiv 0 \tag{1}$$

if β is false.

We consider here only sentences of mathematical nature, i.e., sentences which take values either 0 or 1. For example, the sentence: "two times two is four" is (no doubt) a true one. On the other hand, the statement: "Goethe is a great poet" is not a sentence even though most people would agree that he indeed was a great poet.

Define the sum, the product and negation of sentences and establish the truth tables for each case.

SOLUTION:

If α and β are two sentences, then the sentence "α or β," denoted by $\alpha \vee \beta$, is called the sum of α and β. The sentence "α and β," denoted by $\alpha \wedge \beta$, is called the product of α and β. Clearly, the sentence $\alpha \vee \beta$ is true if at least one of the components is a true proposition. It has $2 \times 2 = 4$ logical possibilities.

Table 1

α	β	$\alpha \vee \beta$
1	1	1
1	0	1
0	1	1
0	0	0

Sentence $\alpha \wedge \beta$ is true if both factors are true sentences.

Table 2

α	β	$\alpha \wedge \beta$
1	1	1
1	0	0
0	1	0
0	0	0

The sum and product of sentences are commutative and associative, i.e.

$$\alpha \vee \beta \equiv \beta \vee \alpha, \quad \alpha \wedge \beta \equiv \beta \wedge \alpha \tag{2}$$

$$\alpha \vee (\beta \vee \gamma) \equiv (\alpha \vee \beta) \vee \gamma, \quad \alpha \wedge (\beta \wedge \gamma) \equiv (\alpha \wedge \beta) \wedge \gamma. \tag{3}$$

The distributive law holds

$$\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma). \tag{4}$$

In the above formulas we used the equivalence sign $\equiv$. The equivalence $\alpha \equiv \beta$ holds, if and only if α and β have the same logical value. The operation of negation of a sentence α denoted by α' (or by $\rceil \alpha$ or $\sim \alpha$) is defined in such a way that if α is true, then α' is false, and if α is false, then α' is true.

Table 3

α	α'
0	1
1	0

From this we obtain the law of double negation

$$\alpha'' \equiv \alpha. \tag{5}$$

● PROBLEM 1-2

Using the logical notation introduced in Problem 1-1, write down two fundamental theorems of Aristotelian logic. Prove that $\alpha \vee 1 \equiv 1$, $\alpha \wedge 0 \equiv 0$.

SOLUTION:

The first theorem is called the law of the excluded middle (principium tertii exclusi) and, in classical logic, is formulated in the following manner:

From two contradictory sentences, one is true.

Let α represent a sentence, then α' is its contradiction. The law can be written as

$$\alpha \vee \alpha' \equiv 1. \tag{1}$$

The second theorem, called the law of contradiction, states: no sentence can

be true simultaneously with its negation. It can be written briefly as follows:

$$\alpha \wedge \alpha' \equiv 0. \tag{2}$$

Let us examine the truth table for the statement $\alpha \vee 1$:

α	$\alpha \vee 1$
1	1
0	1

Hence, $\alpha \vee 1$ is a statement which is always true.

$$\alpha \vee 1 \equiv 1. \tag{3}$$

Similarly, the truth table for $\alpha \wedge 0$ is

α	$\alpha \wedge 0$
1	0
0	0

and

$$\alpha \wedge 0 \equiv 0. \tag{4}$$

● PROBLEM 1-3

So far, we defined three operations on sentences: product, sum and negation. By applying DeMorgan's laws we see that the number of these fundamental operations can be reduced to two.

SOLUTION:

DeMorgan's laws play an important role in mathematical logic. They can be formulated as follows:

$$(\alpha \vee \beta)' \equiv \alpha' \wedge \beta' \tag{1}$$

$$(\alpha \wedge \beta)' \equiv \alpha' \vee \beta'. \tag{2}$$

The first law (1) asserts that, if it is not true that one of the sentences α and β is true, then both of these sentences are false and conversely.

Taking the negation of (1) we obtain:

$$(\alpha \vee \beta)'' \equiv \alpha \vee \beta \equiv (\alpha' \wedge \beta')'. \tag{3}$$

Equation (3) can be treated as a definition of the sum of sentences.

Here, the sum is defined in terms of two operations: product and negation. Hence, we limit the number of fundamental operations to two, product and negation. Similarly, taking the negation of (2) we obtain

$$\alpha \wedge \beta \equiv (\alpha' \vee \beta')'. \tag{4}$$

Now, the product is defined in terms of sum and negation. Again the number of fundamental operations is reduced to two. Product is defined with the aid of sum and negation. It is easy to verify DeMorgan's laws using the truth tables.

Table 1
for $(\alpha \vee \beta)' \equiv \alpha' \wedge \beta'$

α	β	$(\alpha \vee \beta)'$		$\alpha' \wedge \beta'$
1	1	0		0
1	0	0		0
0	1	0	$\equiv$	0
0	0	1		1

Table 2
for $(\alpha \wedge \beta)' \equiv \alpha' \vee \beta'$

α	β	$(\alpha \wedge \beta)'$		$\alpha' \vee \beta'$
1	1	0		0
1	0	1		1
0	1	1	$\equiv$	1
0	0	1		1

● **PROBLEM 1-4**

Generalize DeMorgan's laws for *n* components.

SOLUTION:

DeMorgan's laws state that

$$(\alpha_1 \vee \alpha_2)' \equiv \alpha_1' \wedge \alpha_2' \tag{1}$$

$$(\alpha_1 \wedge \alpha_2)' \equiv \alpha_1' \vee \alpha_2'. \tag{2}$$

We shall generalize (1) for the system of n sentences: $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$$(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-1} \vee \alpha_n)' \equiv [(\alpha_1 \vee \alpha_2 \ldots \vee \alpha_{n-1}) \vee \alpha_n]' \equiv \tag{3}$$

The sum of the sentences is associative. Applying (1) to (3) we find

$$\equiv (\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-1})' \wedge \alpha_n' \equiv$$

$$\equiv [(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-2}) \vee \alpha_{n-1}]' \wedge \alpha_n' \equiv \tag{4}$$

Applying (1) again we obtain

$$(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-1})' \wedge \alpha_n' \equiv (\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-2})' \wedge \alpha'_{n-1} \wedge \alpha'_n . \tag{5}$$

Observe that applying the above procedure to the sentence $(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n)'$, $(n-1)$ times we find

$$(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n)' \equiv \alpha_1' \wedge \alpha_2' \wedge \ldots \wedge \alpha_n'. \tag{6}$$

Now, take the negation of the product, $(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n)'$. Since the product is associative, we have

$$(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n)' \equiv [(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_{n-1}) \wedge \alpha_n]' \equiv \tag{7}$$

Applying (2) to (7) we obtain

$$\equiv (\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_{n-1})' \vee \alpha_n' \equiv \tag{8}$$

This procedure can be repeated again:

$$\equiv [(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_{n-2}) \wedge \alpha_{n-1}]' \vee \alpha_n' \equiv$$

$$\equiv (\alpha_1 \wedge \ldots \wedge \alpha_{n-2})' \vee \alpha_{n-1} \vee \alpha_n' \equiv \tag{9}$$

After $(n-1)$ steps we obtain

$$(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n)' \equiv \alpha_1' \vee \alpha_2' \vee \ldots \vee \alpha_n'. \tag{10}$$

● PROBLEM 1-5

Another important operation on sentences is implication. It is defined by

$$(\alpha \Rightarrow \beta) \equiv (\alpha' \vee \beta) \tag{1}$$

The sign "⇒" is read "implies" or "if ..., then ...".

1. Write the truth table for the implication $\alpha \Rightarrow \beta$.

2. Explain the difference between "if ..., then ..." as defined in a strictly mathematical way by (1) and everyday "if ..., then ...."

3. Show that

$$\text{if } \alpha \Rightarrow \beta \text{ and } \beta \Rightarrow \alpha, \text{ then } \alpha \equiv \beta. \tag{2}$$

SOLUTION:

Let us start with the truth table for $\alpha \Rightarrow \beta$.

Table 1

α	β	$(\alpha \Rightarrow \beta) \equiv (\alpha' \vee \beta)$
1	1	1
1	0	0
0	1	1
0	0	1

1. The implication $\alpha \Rightarrow \beta$ is always true, except when α is true ($\alpha \equiv 1$) and β is false ($\beta \equiv 0$).
2. Implication has properties similar to deduction. There is, however, a small but significant difference. Let α denote the statement "The sun is shining" and let β denote the statement "I am swimming." The statement "If the sun is shining, then I am swimming" makes sense. When is such a statement false?

When the sun is shining, I am not swimming, i.e., $\alpha \wedge \beta' \equiv 1$. But, that is equivalent to

$$1' \equiv 0 \equiv (\alpha \wedge \beta')' \equiv \alpha' \vee \beta \tag{3}$$

in agreement with Table 1. In everyday language a sentence of the form "If α, then β" means that β is true whenever α is true. In such a case, the statement "If two times two is five, then Paris is the capital of Germany" is regarded as nonsense, since both components are false.

The same statement, in terms of formal language, not only makes sense but is also true. To each of the four possibilities shown in Table 1 (for $\alpha \Rightarrow \beta$), the system of formal language assigns either 0 or 1, even though two of the cases ($\alpha \equiv 0$ and $\beta \equiv 0$, $\alpha \equiv 0$ and $\beta \equiv 1$) appear to be pure nonsense in ordinary language.
3. To prove (2), note that

$$(\alpha \Rightarrow \beta) \text{ and } (\beta \Rightarrow \alpha) \equiv (\alpha' \vee \beta) \wedge (\beta' \vee \alpha) \equiv$$

$$\equiv (\alpha' \wedge \beta') \vee (\alpha' \wedge \alpha) \vee (\beta \wedge \beta') \vee (\beta \wedge \alpha) \equiv$$

$$\equiv (\alpha' \wedge \beta') \vee (\alpha \wedge \beta). \tag{4}$$

The sentence $(\alpha' \wedge \beta') \vee (\alpha \wedge \beta)$ is true when both components α and β have the same logical value; i.e., they are both true or both false.

Hence,

$$\text{if } \alpha \Rightarrow \beta \text{ and } \beta \Rightarrow \alpha, \text{ then } \alpha \equiv \beta. \tag{5}$$

Symbols $\Leftrightarrow$ and $\equiv$ may be used interchangeably.

The symbol $\Leftrightarrow$ is read as: "if, and only if," and indicates implication in both directions. Sometimes, instead of if, and only if, we write iff.

Table 2 lists the symbols used so far.

Table 2

Symbol	Meaning
$\alpha, \beta, \gamma, \delta, \ldots$	logical sentences
$'$ or $\sim$ or $\rceil$	negation (not)
$\vee$	sum (or)
$\wedge$	product (and)
$\Rightarrow$	implication (if ..., then ...)
$\Leftrightarrow$	iff (if, and only if)

● PROBLEM 1-6

1. Prove the syllogism law (sometimes called the law of implication):

$$\text{If } \alpha \Rightarrow \beta \text{ and } \beta \Rightarrow \gamma, \text{ then } \alpha \Rightarrow \gamma. \tag{1}$$

2. Prove the law of contraposition, which is the basis of indirect proof (proof by contradiction, reductio ad absurdum):

$$(\alpha \Rightarrow \beta) \Leftrightarrow (\beta' \Rightarrow \alpha'). \tag{2}$$

3. Prove the law of Duns Scotus:

$$\text{If } \alpha \equiv 0, \text{ then } \alpha \Rightarrow \beta. \tag{3}$$

4. Prove the law of Clausius:

$$\text{If } \alpha' \Rightarrow \beta \text{ for each } \beta, \text{ then } \alpha \equiv 1. \tag{4}$$

SOLUTION:

1. Let us write (1) in the form

$$[(\alpha' \vee \beta) \wedge (\beta' \vee \gamma)] \Rightarrow (\alpha' \vee \gamma). \tag{5}$$

Note that the implication $\alpha \Rightarrow \beta$ is always true, except in the case when $\alpha \equiv 1$ and $\beta \equiv 0$. Suppose

$$(\alpha' \vee \beta) \wedge (\beta' \vee \gamma) \equiv 1 \quad \text{and} \quad \alpha' \vee \gamma \equiv 0. \tag{6}$$

From $\alpha' \vee \gamma \equiv 0$, we conclude that $\alpha \equiv 1$ and $\gamma \equiv 0$. Substituting into (6) we obtain

$$(0 \vee \beta) \wedge (\beta' \vee 0) \equiv 1 \quad \text{or} \quad \beta \wedge \beta' \equiv 1 \tag{7}$$

which is a contradiction.

2. $\Rightarrow$ We shall prove that

$$(\alpha \Rightarrow \beta) \Rightarrow (\beta' \Rightarrow \alpha') \tag{8}$$

which is equivalent to

$$(\alpha \Rightarrow \beta)' \vee (\beta' \Rightarrow \alpha') \tag{9}$$

and to

$$(\alpha' \vee \beta)' \vee (\beta \vee \alpha') \tag{10}$$

and to

$$(\alpha \wedge \beta') \vee (\alpha' \vee \beta). \tag{11}$$

The last statement is true for any combintation of α and β.

$\Leftarrow$

$$(\beta' \Rightarrow \alpha') \Rightarrow (\alpha \Rightarrow \beta) \tag{12}$$

or

$$(\beta \vee \alpha')' \vee (\alpha' \vee \beta) \tag{13}$$

or

$$(\beta' \wedge \alpha) \vee (\alpha' \vee \beta) \equiv 1. \tag{14}$$

■

3. Suppose $\alpha \equiv 0$, then $\alpha' \equiv 1$. Therefore

$$(\alpha \Rightarrow \beta) \equiv (\alpha' \vee \beta) \equiv 1. \tag{15}$$

4. If for $\beta \equiv 0$ and $\beta \equiv 1$ we have

$$(\alpha' \Rightarrow \beta) \equiv (\alpha \vee \beta) \equiv 1 \tag{16}$$

then

$$\alpha \equiv 1. \tag{17}$$

Throughout this book we will be using a box (■) to indicate the end of a proof, theorem, or definition.

Note that there is a shorter way to prove (2)

$$(\alpha \Rightarrow \beta) \equiv (\beta' \vee \alpha'). \tag{18}$$

Indeed,

$$(\alpha' \vee \beta) \equiv (\beta'' \vee \alpha'). \tag{19}$$

● PROBLEM 1–7

Prove that:

1. $\alpha \Rightarrow \alpha \vee \beta$ (1)
2. $\alpha \Rightarrow \alpha$ (2)
3. $\alpha \wedge \beta \Rightarrow \alpha \wedge \beta$ (3)
4. $\alpha \wedge \beta \Rightarrow \beta$ (4)

SOLUTION:

1. We shall apply the definition of implication (see Problem 1–5, (1))

$$(\alpha \Rightarrow \beta) \equiv \alpha' \vee \beta. \tag{5}$$

$$(\alpha \Rightarrow \alpha \vee \beta) \equiv \alpha' \vee \alpha \vee \beta \equiv (\alpha' \vee \alpha) \vee \beta \equiv$$

$$\equiv 1 \vee \beta \equiv 1. \tag{6}$$

Here, we applied the law of excluded middle (see Problem 1–2, (1)).

2.
$$(\alpha \Rightarrow \alpha) \equiv \alpha' \vee \alpha \equiv 1. \tag{7}$$

3. In (2) we can replace α by $\alpha \wedge \beta$ to obtain (3).

4.
$$(\alpha \wedge \beta \Rightarrow \beta) \equiv (\alpha \wedge \beta)' \vee \beta \equiv$$

$$\equiv \alpha' \vee \beta' \vee \beta \equiv 1. \tag{8}$$

In (8) we applied DeMorgan's law (See Problem 1–3, (2)).

Note that each of the above problems can be solved "automatically" using the truth tables.

Table

α	β	$\alpha \vee \beta$	$\alpha \wedge \beta$	$\alpha \Rightarrow \alpha \vee \beta$	$\alpha \Rightarrow \alpha$	$\alpha \wedge \beta \Rightarrow \alpha \wedge \beta$	$\alpha \wedge \beta \Rightarrow \beta$
1	1	1	1	1	1	1	1
1	0	1	0	1	1	1	1
0	1	1	0	1	1	1	1
0	0	0	0	1	1	1	1

Now it should be clear why this system is called a formal language.

● PROBLEM 1-8

1. Prove that:

$$(\alpha \Rightarrow \beta) \Rightarrow (\alpha \wedge \gamma \Rightarrow \beta \wedge \gamma). \tag{1}$$

2. Prove that:

$$(\alpha \equiv \beta) \equiv \quad (\alpha \wedge \beta) \vee (\alpha' \wedge \beta'). \tag{2}$$

SOLUTION:

1. From the definition of implication, we conclude that statement (1) is equivalent to

$$(\alpha \Rightarrow \beta)' \ \vee (\alpha \wedge \gamma \Rightarrow \beta \wedge \gamma) \tag{3}$$

which is equivalent to

$$(\alpha' \vee \beta)' \ \vee [(\alpha \wedge \gamma)' \vee (\beta \wedge \gamma)]. \tag{4}$$

From DeMorgan's law, we find that (4) can be replaced by

$$(\alpha \wedge \beta') \ \vee (\alpha' \vee \gamma') \vee (\beta \wedge \gamma). \tag{5}$$

Statement (5) is always true. Indeed, it is true when $\alpha' \equiv 1$ or $\gamma' \equiv 1$. Suppose $\alpha' \equiv 0$ and $\gamma' \equiv 0$, then $\alpha \equiv 1$ and $\gamma \equiv 1$ and for any β, $(1 \wedge \beta') \vee (\beta \wedge 1) \equiv 1$. Thus

$$(\alpha \wedge \beta') \vee (\alpha' \vee \gamma') \vee (\beta \wedge \gamma) \equiv 1. \tag{6}$$

2. Remember that $\equiv$ can be replaced by $\Leftrightarrow$ (See Problem 1-5). Then

$$(\alpha \equiv \beta) \equiv (\alpha \Leftrightarrow \beta) \equiv (\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha) \equiv$$

$$\equiv (\alpha' \vee \beta) \wedge (\beta' \vee \alpha) \equiv (\alpha' \wedge \beta') \vee (\alpha' \wedge \alpha) \vee (\beta \wedge \beta') \vee (\alpha \wedge \beta) \equiv$$

$$\equiv (\alpha' \wedge \beta') \vee (\alpha \wedge \beta). \tag{7}$$

As always, the proofs can be carried out by means of the truth tables.

● PROBLEM 1-9

1. Prove that if $\alpha \Rightarrow \beta$, then

$$\alpha \wedge \beta \equiv \alpha \tag{1}$$

and

$$\alpha \vee \beta \equiv \beta. \tag{2}$$

2. Prove that if $\alpha \Rightarrow \beta$ and $\gamma \Rightarrow \delta$, then

$$\alpha \wedge \gamma \Rightarrow \beta \wedge \delta \tag{3}$$

$$\alpha \vee \gamma \Rightarrow \beta \vee \delta. \tag{4}$$

SOLUTION:

1. Since $\alpha \Rightarrow \beta$, we have

$$\alpha' \vee \beta \equiv 1 \tag{5}$$

and

$$\alpha \equiv \alpha \wedge 1 \equiv \alpha \wedge (\alpha' \vee \beta) \equiv (\alpha \wedge \alpha') \vee (\alpha \wedge \beta) \equiv$$

$$\equiv \alpha \wedge \beta. \tag{6}$$

Similarly, since $\alpha' \vee \beta \equiv 1$ we obtain

$$(\alpha' \vee \beta)' \equiv \alpha \wedge \beta' \equiv 1' \equiv 0 \tag{7}$$

and

$$\beta \equiv \beta \vee 0 \equiv \beta \vee (\alpha \wedge \beta') \equiv \alpha \vee \beta. \tag{8}$$

2. Since $\alpha \Rightarrow \beta$, then for any sentence γ we have

$$\alpha \wedge \gamma \Rightarrow \beta \wedge \gamma. \tag{9}$$

See Problem 1–8, (1).

Similarly, since $\gamma \Rightarrow \delta$, then

$$\beta \wedge \gamma \Rightarrow \beta \wedge \delta. \tag{10}$$

Combining (9) and (10) we obtain, by virture of Problem 1–6,

$$\alpha \wedge \gamma \Rightarrow \beta \wedge \delta. \tag{11}$$

To prove (4), note that

$$(\alpha \Rightarrow \beta) \Rightarrow (\beta' \Rightarrow \alpha') \tag{12}$$

$$(\gamma \Rightarrow \delta) \Rightarrow (\delta' \Rightarrow \gamma'). \tag{13}$$

Applying (11) to (12) and (13), we find

$$\beta' \wedge \delta' \Rightarrow \alpha' \wedge \gamma'. \tag{14}$$

Applying the law of contraposition (Problem 1–6), (2)) to (14) we obtain from (14)

$$(\alpha' \wedge \gamma')' \Rightarrow (\beta' \wedge \delta')' \tag{15}$$

or

$$(\alpha \vee \gamma) \Rightarrow (\beta \vee \delta). \tag{16}$$

● PROBLEM 1–10

Prove the law of absorption:

$$\alpha \equiv \alpha \vee (\alpha \wedge \beta) \equiv \alpha \wedge (\alpha \vee \beta) \tag{1}$$

and its more general version

$$\alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma). \tag{2}$$

SOLUTION:

We have

$$\alpha \wedge (\alpha \vee \beta) \equiv (\alpha \wedge \alpha) \vee (\alpha \wedge \beta) \equiv \alpha \vee (\alpha \wedge \beta). \tag{3}$$

Now we shall show that

$$\alpha \equiv \alpha \vee (\alpha \wedge \beta). \tag{4}$$

Indeed, if $\alpha \equiv 1$, then

$$\alpha \vee (\alpha \wedge \beta) \equiv 1 \vee (1 \wedge \beta) \equiv 1 \equiv \alpha. \tag{5}$$

If $\alpha \equiv 0$, then

$$\alpha \vee (\alpha \wedge \beta) \equiv 0 \vee (0 \wedge \beta) \equiv 0 \equiv \alpha. \tag{6}$$

That completes (1).

To prove (2), note that

$$(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \equiv [\alpha \wedge (\alpha \vee \gamma)] \vee [\beta \wedge (\alpha \vee \gamma)] \equiv \tag{7}$$

By applying (1) we obtain

$$\equiv \alpha \vee [\beta \wedge (\alpha \vee \gamma)] \equiv \alpha \vee [(\alpha \wedge \beta) \vee (\beta \wedge \gamma)] \equiv$$

$$\equiv [\alpha \vee (\alpha \wedge \beta)] \vee (\beta \wedge \gamma) \equiv \tag{8}$$

Again, by applying (1), we obtain from (8)

$$(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \equiv \alpha \vee (\beta \wedge \gamma). \tag{9}$$

● PROBLEM 1-11

The symmetric difference of the sentences α and β is defined by

$$(\alpha \div \beta) \equiv [(\alpha \wedge \beta') \vee (\alpha' \wedge \beta)]. \tag{1}$$

Prove that:

$$\alpha \vee \beta \equiv [(\alpha \div \beta) \div (\alpha \wedge \beta)]. \tag{2}$$

SOLUTION:

The easiest way to prove (2) is to establish the truth table and to compare the corresponding values. The truth table for $\alpha \div \beta$ is:

Table 1

α	β	$\alpha \wedge \beta'$	$\alpha' \wedge \beta$	$\alpha \div \beta$
1	1	0	0	0
1	0	1	0	1
0	1	0	1	1
0	0	0	0	0

Utilizing Table 1, we obtain

Table 2

α	β	$\alpha \vee \beta$	$(\alpha \div \beta) \div (\alpha \wedge \beta)$
1	1	1	1
1	0	1	1
0	1	1	1
0	0	0	0

By comparing the results of the last two columns of Table 2, we find that (2) is true.

We can also carry out the proof of (2) without the use of the truth table.

$$(\alpha \div \beta) \div (\alpha \wedge \beta) \equiv [(\alpha \div \beta) \wedge (\alpha \wedge \beta)'] \vee [(\alpha \div \beta)' \wedge (\alpha \wedge \beta)] \equiv$$

$$\equiv \{[(\alpha \wedge \beta') \vee (\alpha' \wedge \beta)] \wedge (\alpha' \vee \beta')\} \vee$$

$$\vee \{[(\alpha \wedge \beta') \vee (\alpha' \wedge \beta)]' \wedge (\alpha \wedge \beta)\} \equiv$$

$$\equiv [(\alpha' \vee \beta') \wedge (\alpha \wedge \beta')] \vee [(\alpha' \vee \beta') \wedge (\alpha' \vee \beta)] \vee$$

$$\vee [(\alpha \wedge \beta')' \wedge (\alpha' \wedge \beta)' \wedge (\alpha \wedge \beta)] \equiv$$

$$\equiv (\alpha \wedge \beta') \vee (\alpha' \wedge \beta) \vee [(\alpha' \vee \beta) \wedge (\alpha \vee \beta') \wedge (\alpha \wedge \beta)] \equiv$$

$$\equiv (\alpha \wedge \beta') \vee (\alpha' \wedge \beta) \vee (\alpha \wedge \beta) \equiv$$

$$\equiv (\alpha \wedge \beta') \vee [\beta \wedge (\alpha \vee \alpha')] \equiv$$

$$\equiv \beta \vee (\alpha \wedge \beta') \equiv \alpha \vee \beta. \qquad (3)$$

● PROBLEM 1-12

The digital (logic) circuits operate in the binary mode where each input and output voltage is either 1 or 0. This enables us to use Boolean algebra, which differs from ordinary algebra in that Boolean variables and constants are only allowed to have two possible values, 1 or 0. The voltage values in the circuit are predetermined, in the sense that any voltage in the range of 0V to 0.5V corresponds to "0" and any voltage in the range of 2V to 6V corresponds to "1." The values between 0.5V and 2V should not occur in the circuit. Table 1 lists some commonly used terms.

Table 1

Logic 1	Logic 0
True	False
On	Off
High	Low
Yes	No
Closed switch	Open switch

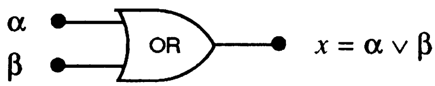
1. Describe the OR gate.

2. Describe the AND gate.

3. Describe the NOT gate (inverter).

SOLUTION:

1. In a digital circuit, an OR gate has two or more inputs, and one output is equal to the sum of the inputs.



α	β	$\alpha \vee \beta = x$
1	1	1
1	0	1
0	1	1
0	0	0

FIGURE 1

Figure 1 shows the symbol of the OR gate with two inputs and its truth table.

An OR gate with 4 inputs is shown in Figure 2.

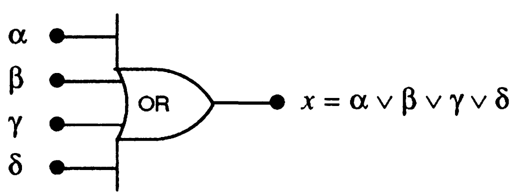
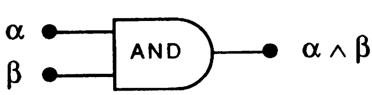


FIGURE 2

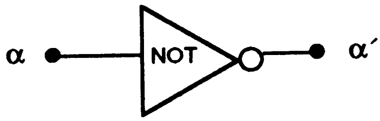
2. A two-input AND gate and its truth table is shown in Figure 3.



α	β	$\alpha \wedge \beta$
1	1	1
1	0	0
0	1	0
0	0	0

FIGURE 3

3. The NOT gate (inverter) has one input and one output.



α	α'
1	0
0	1

FIGURE 4

● PROBLEM 1-13

1. Replace the following digital circuit with the simpler one:

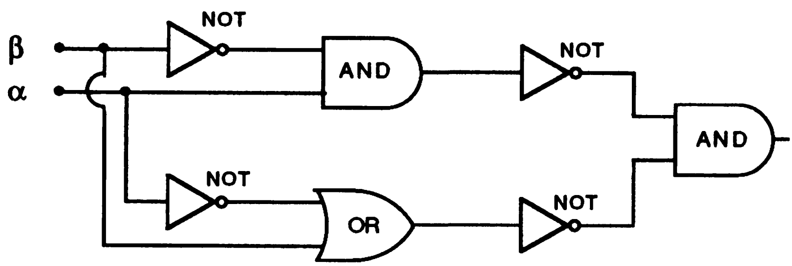


FIGURE 1

2. Illustrate DeMorgan's law using logic gates.

SOLUTION:

1. The circuit shown in Figure 1 has two inputs, one output, and it performs the function described by

$$(\beta' \wedge \alpha)' \wedge (\alpha' \vee \beta)'. \tag{1}$$

By applying DeMorgan's laws and some basic properties of sentence calculus, we obtain

$$(\beta' \wedge \alpha)' \wedge (\alpha' \vee \beta)' \equiv (\beta \vee \alpha') \wedge (\alpha \wedge \beta') \equiv$$

$$\equiv [(\alpha \wedge \beta') \wedge \beta] \vee [(\alpha \wedge \beta') \wedge \alpha'] \equiv$$

$$\equiv (\alpha \wedge \beta' \wedge \beta) \vee (\alpha \wedge \alpha' \wedge \beta') \equiv$$

$$\equiv (\alpha \wedge 0) \vee (0 \wedge \beta') \equiv 0 \vee 0 \equiv 0. \tag{2}$$

No matter what the input, the output is always 0. To construct a circuit equivalent to the one shown in Figure 1, all that is needed is wire. See Figure 2.

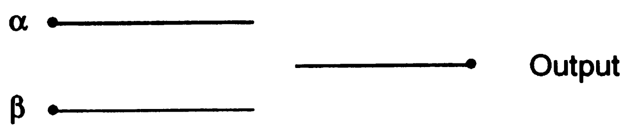
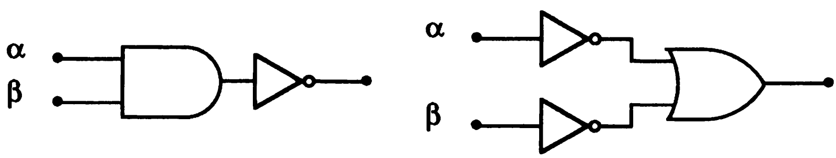


FIGURE 2

Luckily, not all of the computer logic circuits can be simplified to such an extent.

2. We will build the circuit for

$$(\alpha \wedge \beta)' \equiv \alpha' \vee \beta'. \tag{3}$$



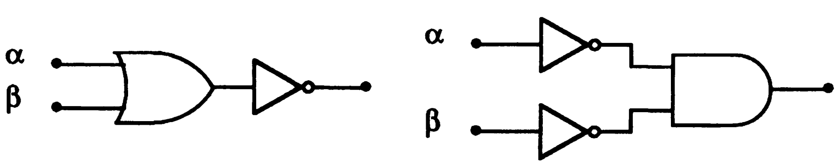
(a) (b)

FIGURE 3

Circuits (a) and (b) shown in Figure 3 are equivalent.

Now we will build the circuit for

$$(\alpha \vee \beta)' \equiv \alpha' \wedge \beta'. \tag{4}$$



(a) (b)

FIGURE 4

Circuits (a) and (b) of Figure 4 are equivalent.

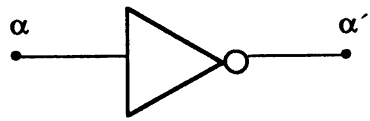
Design the circuit that implements the expression

1. $$[(\alpha' \wedge \beta) \vee (\beta' \wedge \gamma)]'. \quad (1)$$

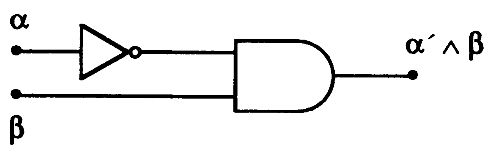
2. $$[(\alpha \wedge \beta \wedge \gamma) \vee (\alpha' \wedge \beta')] \wedge \gamma'. \quad (2)$$

SOLUTION:

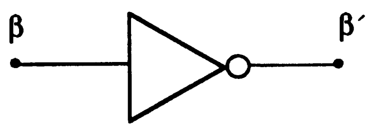
1. Step by step we shall move α', which is realized by



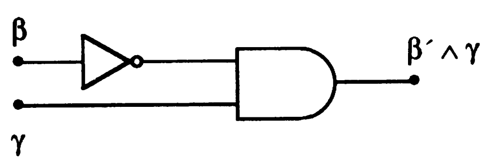
$\alpha' \wedge \beta$ is realized by



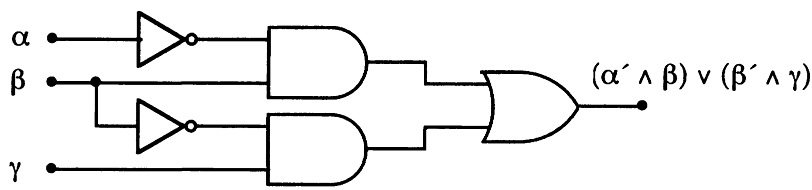
β' is realized by



$\beta' \wedge \gamma$ is realized by



$(\alpha' \wedge \beta) \vee (\beta' \wedge \gamma)$ is realized by



$[(\alpha' \wedge \beta) \vee (\beta' \wedge \gamma)]'$ is realized by

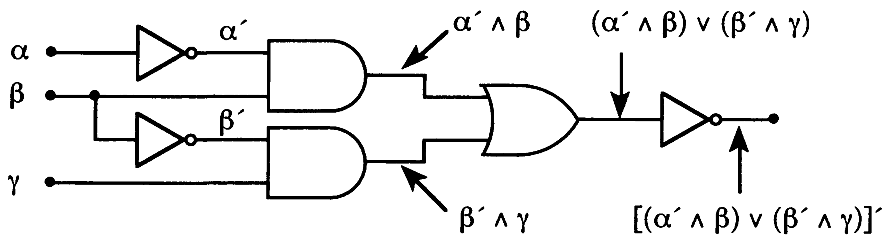
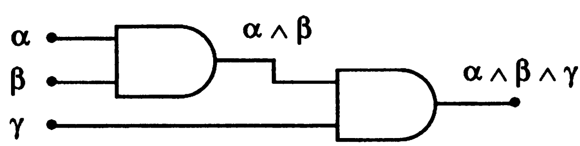
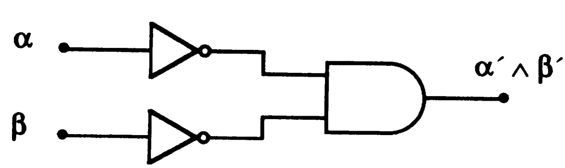


FIGURE 1

2. $\alpha \wedge \beta \wedge \gamma$ is implemented by



$\alpha' \wedge \beta'$ is implemented by



$[(\alpha \wedge \beta \wedge \gamma) \vee (\alpha' \wedge \beta')] \wedge \gamma'$ is implemented by

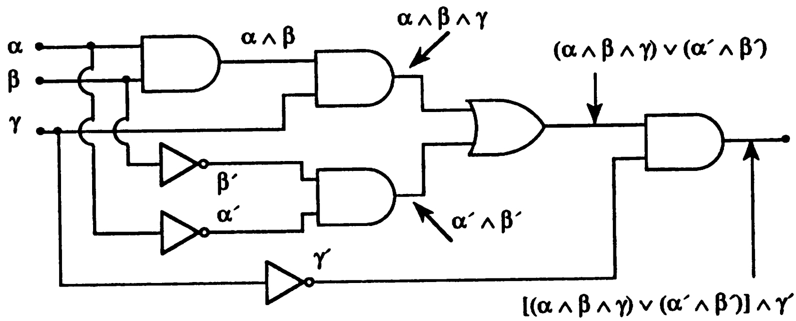


FIGURE 2

Note that since

$$[(\alpha \wedge \beta \wedge \gamma) \vee (\alpha' \wedge \beta')] \wedge \gamma' \equiv$$

$$\equiv (\alpha \wedge \beta \wedge \gamma \wedge \gamma') \vee (\alpha' \wedge \beta' \wedge \gamma') \equiv (\alpha' \wedge \beta') \wedge \gamma' \tag{3}$$

the circuit in the last part of Figure 2 can be replaced by

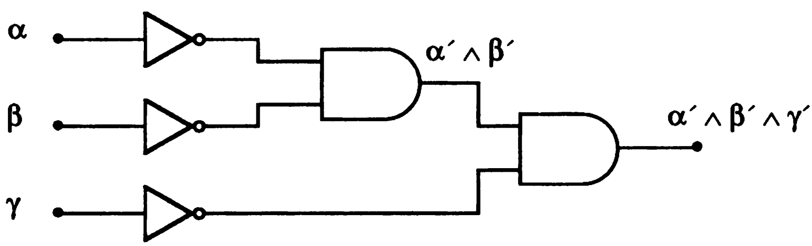


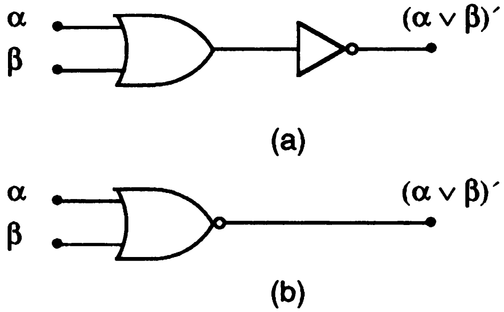
FIGURE 3

● PROBLEM 1-15

> 1. Define NOR and NAND gates.
>
> 2. Show that any logic circuit can be built exclusively with the NAND gates.

SOLUTION:

1. NOR gate consists of OR gate and an INVERTER.

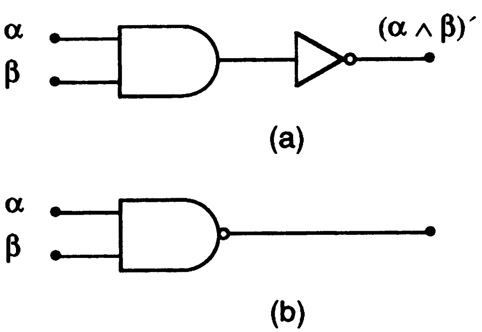


α	β	$\alpha \vee \beta$	$(\alpha \vee \beta)'$
1	1	1	0
0	1	1	0
1	0	1	0
0	0	0	1

FIGURE 1—NOR Gate

Both figures represent the same logic circuit. The symbol in Figure (b) is used for convenience.

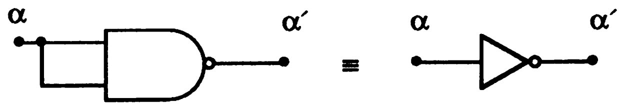
NAND gate and its symbol are shown in Figure 2.



α	β	$\alpha \wedge \beta$	$(\alpha \wedge \beta)'$
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	1

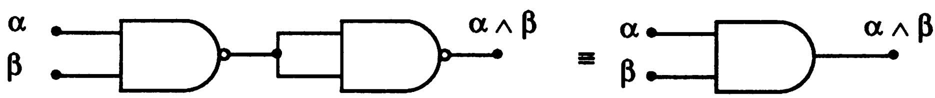
FIGURE 2—NAND Gate

2. All Boolean expressions consist of various combinations of the basic operations of OR, AND, and INVERT. Thus, any circiut can be implemented using OR, AND, and INVERT gates. It is possible, however, to build any logic circuit using only NAND gates. Using NAND gates, we can design any logic operations OR, AND, and INVERT.



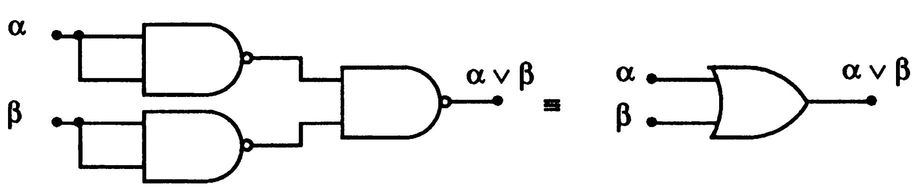
(a) INVERTER

Indeed $$(\alpha \wedge \alpha)' \equiv \alpha'$$



(b) AND Gate

Indeed $$[(\alpha \wedge \beta)' \wedge (\alpha \wedge \beta)']' \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \beta) \equiv \alpha \wedge \beta\,.$$



(c) OR Gate

Indeed $$[(\alpha \wedge \alpha)' \wedge (\beta \wedge \beta)']' \equiv (\alpha' \wedge \beta')' \equiv \alpha \vee \beta\,.$$

FIGURE 3

NAND gates can be used to perform any Boolean function.

1. Design a logic circuit that implements the function

$$(\alpha \wedge \beta) \vee (\gamma \wedge \delta). \tag{1}$$

2. Suppose you have the TTL IC's shown in Figure 1 at your disposal. Each IC is a quad; that is, each chip contains four identical two-input gates. Using the minimum number of IC's, design the circuit which performs function (1).

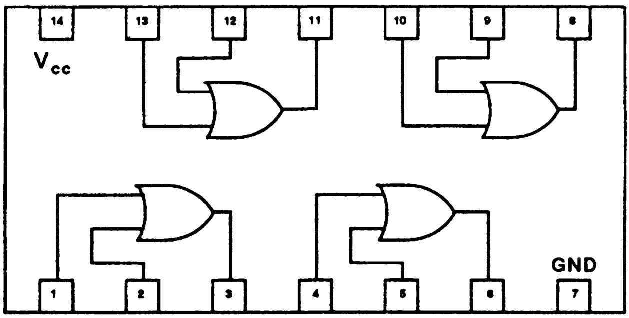


FIGURE 1a — IC 7432

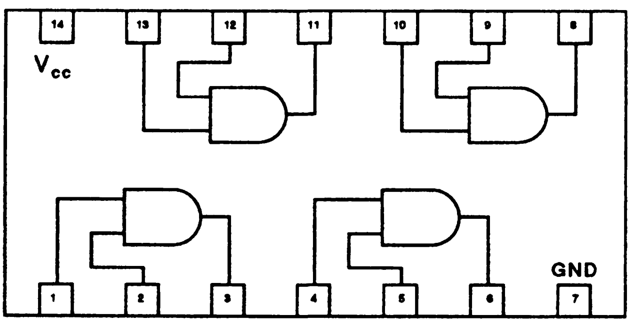


FIGURE 1b — IC 7408

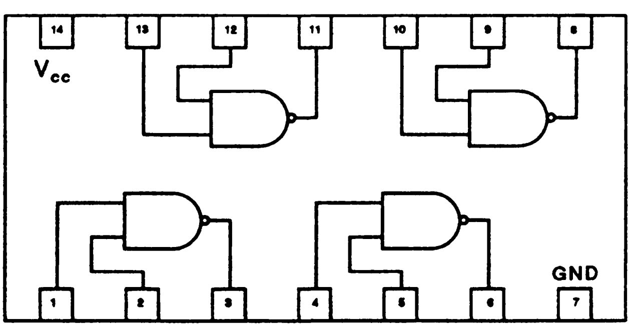


FIGURE 1c — IC 7400

SOLUTION:

1.

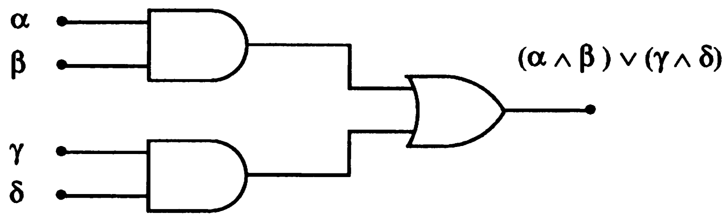


FIGURE 2

Figure 2 shows the circuit implementating operation $(\alpha \wedge \beta) \vee (\gamma \wedge \delta)$.

2. From Figure 2 we conclude that in order to build the circuit (1), we have to use two gates from IC 7408 and one gate from IC 7432, as shown in Figure 3.

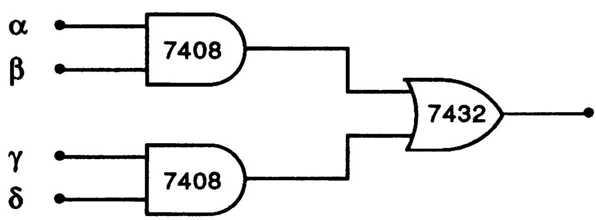


FIGURE 3

Each AND gate and OR gate can be replaced by a combination of NAND gates (See Problem 1–15).

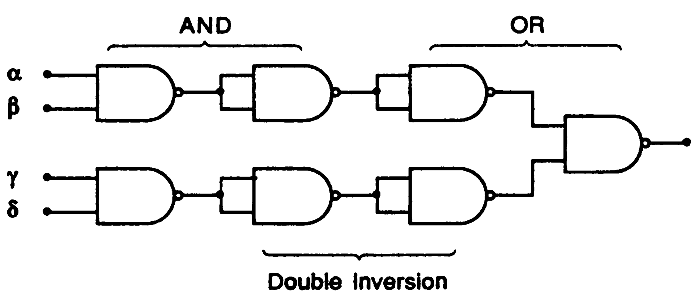


FIGURE 4

By eliminating double inversion, we obtain

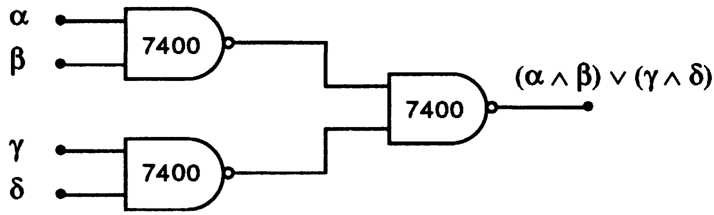
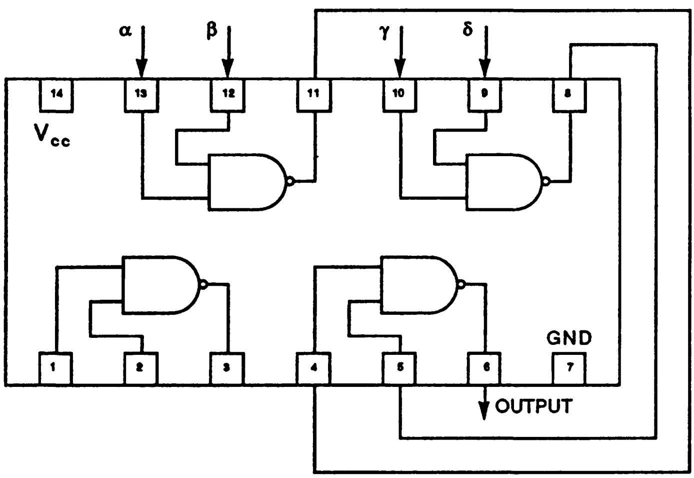


FIGURE 5

Now to implement the operation, we only need three NAND gates. Therefore, instead of using two IC's as in Figure 3, we can use one IC 7400, as shown in Figure 5 and Figure 6.



IC 7400

$$\text{OUTPUT} = (\alpha \wedge \beta) \vee (\gamma \wedge \delta)$$

FIGURE 6

CHAPTER 2

ALGEBRA OF SETS

Which of the following statements is true?

1. $7 \in A$, where

$$A = \{x : x \in N \text{ and } 5 < x < 8\}$$

and N is the set of natural numbers.

2. $B = 4$, where

$$B = \{x : x + 1 = 5\}.$$

SOLUTION:

The theory of sets was founded and developed into a mathematical system by G. Cantor (1845-1918). According to him, we are to understand by a set — a collection into a whole, of definite, well-distinguished objects (called the elements of A) of our perception or of our thought."

For example, the set of vertices of a square consists of four elements. The set can be defined by listing its elements.

For example:

$$A = \{\text{John, seven}\}$$

is the set consisting of two elements: John and seven. The statement "x is an element of A" is written

$$x \in A.$$

The negation of the statement $x \in A$ is

$$x \notin A.$$

The other way to define a set is to characterize the properties (or property) of its elements.

$$A = \{x : p(x)\}.$$

If $x \in A$, then $p(x)$ is true ($p(x) \equiv 1$).

For example:

$$N = \{n : n \text{ is a natural number}\}.$$

1. The set A consists of two elements

$$A = \{6, 7\}.$$

Thus

$$7 \in A.$$

2. The set B consists of a single element

$$B = \{4\}.$$

The number 4 belongs to the set B, $4 \in B$, but it does not equal B. The statement $B = 4$ is not true.

● PROBLEM 2-2

Show that equality of sets is reflexive, symmetric, and transitive, i.e., show that

1. $A = A$ for all sets A.
2. If $A = B$, then $B = A$ for all sets A, B.
3. If $A = B$ and $B = C$, then $A = C$ for all sets A, B, C.

SOLUTION:

Two sets, A and B, are said to be equal, $A = B$, provided that they contain the same elements. From this definition we conclude that:

1. $x \in A$ if and only if $x \in A$, thus

$$A = A.$$

Frequently, instead of writing if, and only if, we shall use iff.

"iff" = if and only if.

2. The statement $x \in A$ if and only if (iff) $x \in B$ is equivalent to $x \in B$, iff $x \in A$.

3. ($x \in A$ if and only if $x \in B$) and ($x \in B$ if and only if $x \in C$) imply ($x \in A$ if and only if $x \in C$). Using the logic notation, we can write this statement in the form

$$[(x \in A \text{ iff } x \in B) \wedge (x \in B \text{ iff } x \in C)] \Rightarrow (x \in A \text{ iff } x \in C)$$

or

$$(A = B) \wedge (B = C) \Rightarrow (A = C).$$

1. Show that inclusion of sets is reflexive, anti-symmetric, and transitive; i.e.

   a. $A \subset A$ for all sets A.

   b. $A \subset B$ and $B \subset A$ imply $A = B$ for all A, B.

   c. If $A \subset B$ and $B \subset C$, then $A \subset C$ for all sets $A, B,$ and C.

2. Prove

$$[(A \subset B) \wedge (B \subset C) \wedge (C \subset A)] \Rightarrow (A = B = C). \tag{1}$$

SOLUTION:

DEFINITION: INCLUSION

The set A is included in B; i.e., A is a subset of B, which is written as $A \subset B$ or $B \subset A$, when

$$x \in A \Rightarrow x \in B \tag{2}$$

■

1. Obviously

$$(x \in A) \Rightarrow (x \in A) \text{ for all } A. \tag{3}$$

Hence
$$A \subset A.$$

$$(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A) \Rightarrow$$

$$(x \in A \text{ iff } x \in B). \tag{4}$$

Hence
$$(A \subset B) \wedge (B \subset A) \Rightarrow A = B. \tag{5}$$

$$(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in C) \Rightarrow$$

$$(x \in A \Rightarrow x \in C). \tag{6}$$

Thus

$$(A \subset B) \wedge (B \subset C) \Rightarrow A \subset C. \tag{7}$$

2. To prove that $A = B$, we must prove that $x \in A$ iff $x \in B$. Since $A \subset B$, it is

enough to show that $B \subset A$. But $B \subset C$ and $C \subset A$, thus $B \subset A$. And

$$A \subset B \text{ and } B \subset A \Rightarrow A = B. \tag{8}$$

Similarly, we show that $B = C$.

$$(C \subset A \text{ and } A \subset B) \Rightarrow C \subset B. \tag{9}$$

Thus

$$(B \subset C \text{ and } C \subset B) \Rightarrow C = B.$$

● PROBLEM 2-4

Using Venn diagrams, illustrate the union, intersection and difference of two sets A and B.

Show that intersection is associative; i.e., that

$$A \cap (B \cap C) = (A \cap B) \cap C \tag{1}$$

for all sets A, B, and C.

SOLUTION:

The union of two sets A and B is the set whose elements are all the elements of the set A and all the elements of the set B and which does not contain any other elements. The union of A and B is denoted by $A \cup B$; see Figure 1.

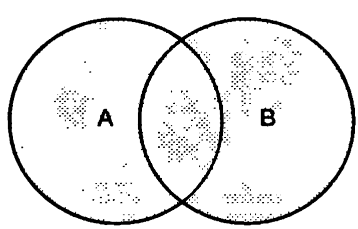


FIGURE 1. A ∪ B is shaded.

The common part of two sets is called the intersection. The intersection of A and B contains those, and only those, elements which belong to A and to B. The intersection is denoted by $A \cap B$.

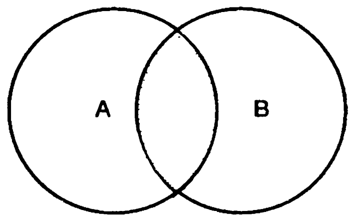


FIGURE 2. $\mathbf{A \cap B}$ **is shaded.**

The difference of two sets A and B is the set consisting of those, and only those, elements which belong to A, but do not belong to B. The difference is denoted by $A - B$ (sometimes $A \setminus B$).

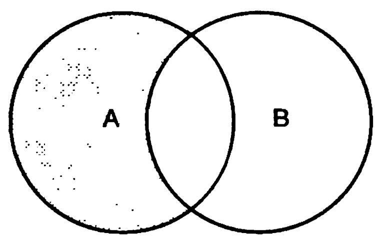


FIGURE 3. $\mathbf{A - B}$ **is shaded.**

Now we will use Venn diagrams to prove (1).

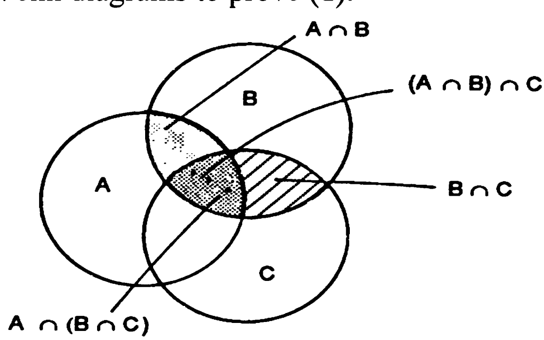


FIGURE 4.

From Figure 4 we conclude that

$$A \cap (B \cap C) = (A \cap B) \cap C.$$

● PROBLEM 2-5

By applying the sentence calculus, prove the following formula:

$$A \cup (A \cap B) = A = A \cap (A \cup B). \tag{1}$$

SOLUTION:

Operations on sets are related to operations on sentences. To denote that x is an element of A we write

$$x \in A. \tag{2}$$

Similarly

$$(x \notin A) \equiv (x \in A)' \tag{3}$$

where $x \notin A$ means x is not an element of A.

The following equivalences hold for all x:

$$(x \in A \cup B) \equiv (x \in A) \vee (x \in B) \tag{4}$$

$$(x \in A \cap B) \equiv (x \in A) \wedge (x \in B) \tag{5}$$

$$(x \in A - B) \equiv (x \in A) \wedge (x \in B)' \tag{6}$$

Using formulas (3) - (6) we can deduce theorems on the calculus of sets from analogous theorems of sentence calculus. Furthermore, let us note that

$$\text{if } x \in A \equiv x \in B \text{ holds for all } x \text{, then } A = B. \tag{7}$$

We shall now prove (1).

$$[x \in A \cup (A \cap B)] \equiv (x \in A) \vee (x \in A \cap B) \equiv$$

$$\equiv (x \in A) \vee [(x \in A) \wedge (x \in B)] \equiv (x \in A). \tag{8}$$

Thus

$$A \cup (A \cap B) = A. \tag{9}$$

Similarly

$$[x \in A \cap (A \cup B)] \equiv (x \in A) \wedge (x \in A \cup B) \equiv$$

$$\equiv (x \in A) \wedge [(x \in A) \vee (x \in B)] \equiv (x \in A). \tag{10}$$

Hence

$$A \cap (A \cup B) = A. \tag{11}$$

Note that in (8) and (10) we applied the law of absorption of sentence calculus.

$$\alpha \vee (\alpha \wedge \beta) \equiv \alpha \equiv \alpha \wedge (\alpha \vee \beta) \tag{12}$$

● PROBLEM 2-6

Prove these basic formulas for unions and intersections:

1. $\left.\begin{array}{l} A \cup A = A \\ A \cap A = A \end{array}\right\}$ indempotency (1)

2. $\left.\begin{array}{l} A \cup B = B \cup A \\ A \cap B = B \cap A \end{array}\right\}$ commutativity (2)

3. $\left.\begin{array}{l} A \cup (B \cup C) = (A \cup B) \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C \end{array}\right\}$ associativity (3)

4. $\left.\begin{array}{l} A \cup (A \cap B) = A \\ A \cap (A \cup B) = A \end{array}\right\}$ adjunction (4)

5. $\left.\begin{array}{l} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{array}\right\}$ distributivity (5)

SOLUTION:

1.
$$(x \in A \cup A) \equiv (x \in A) \vee (x \in A) \equiv (x \in A) \tag{6}$$

Hence

$$A \cup A = A. \tag{7}$$

$$(x \in A \cap A) \equiv (x \in A) \wedge (x \in A) \equiv (x \in A) \tag{8}$$

Hence

$$A \cap A = A. \tag{9}$$

From (7) and (9) we see that, in contrast to arithmetic, neither multiples nor exponents arise in set theory.

2.
$$(x \in A \cup B) \equiv (x \in A) \vee (x \in B) \equiv$$
$$\equiv (x \in B) \vee (x \in A) \equiv (x \in B \cup A). \tag{10}$$

Thus
$$A \cup B = B \cup A. \tag{11}$$

$$(x \in A \cap B) \equiv (x \in A) \wedge (x \in B) \equiv$$
$$\equiv (x \in B) \wedge (x \in A) \equiv (x \in B \cap A). \tag{12}$$

Hence
$$A \cap B = B \cap A. \tag{13}$$

3.
$$[x \in A \cup (B \cup C)] \equiv [(x \in A) \vee (x \in B \cup C)] \equiv$$
$$\equiv [(x \in A) \vee (x \in B) \vee (x \in C)] \equiv [(x \in A \cup B) \vee (x \in C)] \equiv \tag{14}$$
$$\equiv [x \in (A \cup B) \cup C]$$

Thus
$$A \cup (B \cup C) = (A \cup B) \cup C. \tag{15}$$

Replacing "$\vee$" by "$\wedge$" in (14) we prove
$$A \cap (B \cap C) = (A \cap B) \cap C. \tag{16}$$

4. See Problem 2–5.

5.
$$[x \in A \cup (B \cap C)] \equiv [(x \in A) \vee (x \in B \cap C)] \equiv$$
$$\equiv \{(x \in A) \vee [(x \in B) \wedge (x \in C)]\} \equiv$$
$$\equiv \{[(x \in A) \vee (x \in B)] \wedge [(x \in A) \vee (x \in C)]\} \equiv$$
$$\equiv (x \in A \cup B) \wedge (x \in A \cup C) \equiv$$
$$\equiv [x \in (A \cup B) \cap (A \cup C)]. \tag{17}$$

Hence
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \tag{18}$$

$$[x \in A \cap (B \cup C)] \equiv [(x \in A) \wedge (x \in B \cup C)] \equiv$$
$$\equiv \{(x \in A) \wedge [(x \in B) \vee (x \in C)]\} \equiv$$

$$\equiv \{[(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)]\} \equiv$$

$$\equiv [(x \in A \cap B) \vee (x \in A \cap C)] \equiv$$

$$\equiv [x \in (A \cap B) \cup (A \cap C)]\,. \tag{19}$$

Hence $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{20}$$

● PROBLEM 2-7

1. Find the elements of the set

$$A = \{x : x + 1 = 3, x^2 = 1\}. \tag{1}$$

2. Prove that:

If A is a subset of the null set ϕ, then $A = \phi$.

3. Prove that:

$$(A - B) \cap B = \phi. \tag{2}$$

SOLUTION:

DEFINITION OF THE NULL SET

The null set (or the empty set or the void set) is the set which contains no elements. The null set is denoted by ϕ. Often the null set is defined as

$$\phi = \{x : x \neq x\}. \tag{3}$$

1. Number x, such that $x + 1 = 3$ and $x^2 = 1$ does not exist, hence

$$A = \phi. \tag{4}$$

2. The null set ϕ is a subset of every set, thus $\phi \subset A$. But, by hypothesis, $A \subset \phi$, therefore

$$A = \phi. \tag{5}$$

3. $$[x \in (A - B) \cap B] \equiv$$

$$\equiv [(x \in A - B) \wedge (x \in B)] \equiv$$

$$\equiv [(x \in A) \wedge (x \in B)' \wedge (x \in B)] \equiv 0. \tag{6}$$

Thus, the set $(A - B) \cap B$ contains no elements

$$(A - B) \cap B = \phi. \tag{7}$$

● PROBLEM 2-8

1. Find the power set $P(A)$ of the set

$$A = \{1, 2, 3\}. \tag{1}$$

2. Find the power set $P(B)$ of the set

$$B = \{1, \{2, 3\}\}. \tag{2}$$

3. Set A consists of 10 elements. How many elements does the power set $P(A)$ have?

SOLUTION:

DEFINITION OF POWER SET

The power set $P(A)$ of A is the class of all subsets of A.

■

1. The subsets of A are:

$$\begin{array}{ll} \text{0 elements:} & \phi \\ \text{1 element:} & \{1\}, \{2\}, \{3\} \\ \text{2 elements:} & \{1, 2\}, \{1, 3\}, \{2, 3\} \\ \text{3 elements:} & \{1, 2, 3\} \end{array} \tag{3}$$

Hence

$$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1, 2, 3\}\} \tag{4}$$

$P(A)$ contains $2^3 = 8$ elements.

2. Note that B contains only two elements, 1 and $\{2, 3\}$. Thus, its power set $P(B)$ is

$$P(B) = \{\phi, \{1\}, \{2,3\}, \{1, \{2, 3\}\}\}. \tag{5}$$

3. From the theory of permutations we have the equation for the number of different combinations that can be formed from n different elements, using m elements at a time.

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \tag{6}$$

where

$$n! = 1 \cdot 2 \cdot \ \ldots \ \cdot (n-1) \cdot n \tag{7}$$

and

$$0! = 1.$$

Suppose set A contains n elements; then there are

$\binom{n}{0} = 1$ subsets with 0 elements

$\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$ subsets with 1 element

$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$ subsets with 2 elements.

The total number of subsets of a set A consisting of n elements is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = (1+1)^n = 2^n. \tag{8}$$

Set A consists of 10 elements, hence its power set $P(A)$ consists of $2^{10} = 1024$.

● PROBLEM 2-9

If A is any non-empty set, then $P(A)$ (or 2^A) is the power set of A. Prove that:

1. $2^A \cap 2^B = 2^{A \cap B}$ (1)

2. $2^A \cup 2^B \subset 2^{A \cup B}$ (2)

3. Show that $2^A \cup 2^B \neq 2^{A \cup B}$ (3)

SOLUTION:

1. Let α represent an element of $2^A \cap 2^B$. Recall that the power set 2^A of A is the class of all subsets of A. Suppose α consists of elements $p_1, p_2, \ldots$

$$\alpha = \{p_1, p_2, \ldots\} \tag{4}$$

Since $\alpha \in 2^A$ and $\alpha \in 2^B$, each of $p_1, p_2, \ldots$ belongs to $A \cap B$. Thus, α is a subset of $A \cap B$ and

$$\alpha \in 2^{A \cap B} \tag{5}$$

Hence,

$$2^A \cap 2^B \subset 2^{A \cap B}. \tag{6}$$

Now, suppose $\alpha \in 2^{A \cap B}$; i.e., α is a subset of $A \cap B$, and each element of $\alpha = \{p_1, p_2, \ldots\}$ belongs to $A \cap B$. Hence, each of $p_1, p_2, \ldots$ belongs to A and to B.

Therefore, α is a subset of A, $\alpha \in 2^A$; and α is a subset of B, $\alpha \in 2^B$.

$$\alpha \in 2^A \cap 2^B. \tag{7}$$

Hence,

$$2^{A \cap B} \subset 2^A \cap 2^B. \tag{8}$$

From (6) and (8) we conclude that

$$2^{A \cap B} = 2^A \cap 2^B. \tag{9}$$

2. Suppose $\alpha \in 2^A \cup 2^B$; that is, $\alpha \in 2^A$ or $\alpha \in 2^B$. Assume $\alpha \in 2^A$, then α is a subset of A. If α is a subset of A, then α is a subset of $A \cup B$ and

$$\alpha \in 2^{A \cup B}. \tag{10}$$

That proves

$$2^A \cup 2^B \subset 2^{A \cup B}. \tag{11}$$

3. Now we shall show that

$$2^A \cup 2^B \neq 2^{A \cup B}. \tag{12}$$

Let $A = \{a\}$, $B = \{b\}$, then

$$2^A = \{\phi, \{a\}\} \tag{13}$$

$$2^B = \{\phi, \{b\}\} \tag{14}$$

and $A \cup B = \{a, b\}$

$$2^{A \cup B} = \{\phi, \{a\}, \{b\}, \{a, b\}\}. \tag{14}$$

Hence,

$$2^A \cup 2^B = \{\phi, \{a\}, \{b\}\} \tag{15}$$

and

$$2^A \cup 2^B \neq 2^{A \cup B}. \tag{16}$$

● PROBLEM 2–10

Prove the following formulas:

1. $A \cap B \subset A \subset A \cup B$ (1)

2. $A - B \subset A$ (2)

3. If $A \subset B$ and $C \subset D$, then

$$A \cup C \subset B \cup D \tag{3}$$

and

$$A \cap C \subset B \cap D \tag{4}$$

4. $(A \subset B) \equiv (A \cup B = B) \equiv (A \cap B = A)$ (5)

5. $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$ (6)

SOLUTION:

1. $[x \in A \cap B] \equiv [(x \in A) \wedge (x \in B)] \Rightarrow (x \in A)$ (7)

Thus,

$$A \cap B \subset A \tag{8}$$

$$(x \in A)\ [(x \in A) \vee (x \in B)] \equiv (x \in A \cup B) \tag{9}$$

Hence,

$$A \subset A \cup B. \tag{10}$$

2. $[x \in A - B] \equiv [(x \in A) \vee (x \in B)'] \Rightarrow (x \in A)$ (11)

Hence,

$$A - B \subset A. \tag{12}$$

3. We have to show that

$$[(A \subset B) \wedge (C \subset D)] \Rightarrow \begin{pmatrix} A \cup C \subset B \cup D \\ \text{and} \\ A \cap C \subset B \cap D \end{pmatrix} \tag{13}$$

$$[x \in A \cup C] \equiv [(x \in A) \vee (x \in C)] \Rightarrow$$

$$\Rightarrow [(x \in B) \vee (x \in D)] \equiv (x \in B \cup D) \tag{14}$$

Hence,

$$A \cup C \subset B \cup D. \tag{15}$$

$$(x \in A \cap C) \equiv [(x \in A) \wedge (x \in C)] \Rightarrow$$

$$\Rightarrow [(x \in B) \wedge (x \in D)] \equiv (x \in B \cap D) \tag{16}$$

Hence,

$$A \cap C \subset B \cap D. \tag{17}$$

4. Suppose $A \subset B$; then from $(A \subset B)$ and $(B \subset B)$ we conclude that

$$(A \cup B) \subset (B \cup B) = B \tag{18}$$

by virtue of (13). But since $B \subset (A \cup B)$, we have

$$A \cup B = B. \tag{19}$$

Conversely from $A \cup B = B$, it follows that $A \subset B$. Hence, the relations $A \subset B$ and $A \cup B = B$ are equivalent.

$$(A \subset B) \equiv (A \cup B = B). \tag{20}$$

Similarly, combining $A \subset B$ with $A \subset A$, we obtain $A \subset (A \cap B)$. Since $A \cap B \subset A$,

$$A = A \cap B. \tag{21}$$

Conversely from $A = A \cap B$ we conclude that $A \subset B$.

5. $$(A \cup B) \cap (A \cup C) = (A \cap A) \cup (A \cap C) \cup (B \cap A) \cup (B \cap C) =$$

$$= A \cup (A \cap C) \cup (A \cap B) \cup (B \cap C) \tag{22}$$

But $A \cap B \subset A$, hence, $A \cup (A \cap B) = A$ and $A \cap C \subset A$, hence, $A \cup (A \cap C) = A$. Thus,

$$(A \cup B) \cap (A \cup C) = A \cup (B \cap C). \tag{23}$$

Prove the following formulas:

1. $A \cap B = A - (A - B)$ (1)

2. $A \cup (B - A) = A \cup B$ (2)

3. $A - (A \cap B) = A - B$ (3)

4. $A \cap (B - C) = (A \cap B) - (A \cap C)$ (4)

5. $A \cup (B - C) \neq (A \cup B) - (A \cup C)$ (5)

SOLUTION:

1.
$$\begin{aligned}[x \in A - (A - B)] &\equiv [(x \in A) \wedge (x \in A - B)'] \equiv \\ &\equiv \{(x \in A) \wedge [(x \in A) \wedge (x \in B)']'\} \equiv \\ &\equiv \{(x \in A) \wedge [(x \in A)' \vee (x \in B)]\} \equiv \\ &\equiv [(x \in A) \wedge (x \in B)] \equiv (x \in A \cap B). \end{aligned} \tag{6}$$

2.
$$\begin{aligned}[x \in A \cup (B - A)] &\equiv [(x \in A) \vee (x \in B - A)] \equiv \\ &\equiv \{(x \in A) \vee [(x \in B) \wedge (x \in A)']\} \equiv \\ &\equiv \{[(x \in A) \vee (x \in B)] \wedge [(x \in A) \vee (x \in A)']\} \equiv \\ &\equiv [(x \in A) \vee (x \in B)] \equiv (x \in A \cup B). \end{aligned} \tag{7}$$

Hence,

$$A \cup (B - A) = A \cup B. \tag{8}$$

3.
$$\begin{aligned}[x \in A - (A \cap B)] &\equiv [(x \in A) \wedge (x \in A \cap B)'] \equiv \\ &\equiv \{(x \in A) \wedge [(x \in A) \wedge (x \in B)]'\} \equiv \\ &\equiv \{(x \in A) \wedge [(x \in A)' \vee (x \in B)']\} \equiv \end{aligned}$$

$$\equiv \{[(x \in A) \wedge (x \in A)'] \vee [(x \in A) \wedge (x \in B)']\} \equiv$$

$$\equiv [(x \in A) \wedge (x \in B)'] \equiv (x \in A - B). \tag{9}$$

Hence,

$$A - (A \cap B) = A - B. \tag{10}$$

4. $$[x \in (A \cap B) - (A \cap C)] \equiv [(x \in A \cap B) \wedge (x \in A \cap C)'] \equiv$$

$$\equiv \{(x \in A) \wedge (x \in B) \wedge [(x \in A) \wedge (x \in C)]'\} \equiv$$

$$\equiv \{(x \in A) \wedge (x \in B) \wedge [(x \in A)' \vee (x \in C)']\} \equiv$$

$$\equiv [(x \in A) \wedge (x \in B) \wedge (x \in C)'] \equiv$$

$$\equiv [(x \in A) \wedge (x \in B - C)] \equiv [x \in A \cap (B - C)]. \tag{11}$$

5. Suppose A, B, C are three disjoint sets as shown in the Venn diagram:

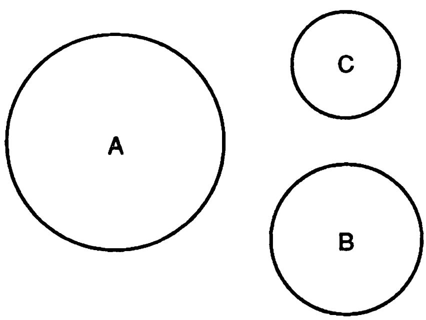


FIGURE 1

Then

$$A \cup (B - C) = A \cup B \tag{12}$$

and

$$(A \cup B) - (A \cup C) = B \tag{13}$$

Hence,

$$A \cup (B - C) \neq (A \cup B) - (A \cup C). \tag{14}$$

In the theory of sets, we assume that all the sets under consideration are subsets of some fixed set, called the space (or universal set). For example, in geometry the space is the Euclidean space, and in analysis the space is the set of real or complex numbers.

Throughout this book, all the sets considered will belong to the space which we shall denote by (1). Hence

$$A \subset 1 \tag{1}$$

for each of the sets considered.

DEFINITION OF COMPLEMENT OF THE SET

The complement of the set A with respect to the given space 1 is denoted by A^C (or CA or $\sim A$) and defined by

$$A^C = 1 - A \tag{2}$$

See Figure 1.

■

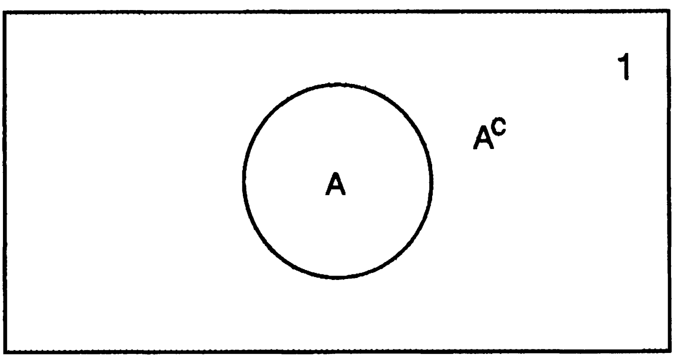


FIGURE 1.

We have

$$(x \in A^C) \equiv (x \in A)' \equiv (x \notin A) \tag{3}$$

Prove that:

$$1^C = \phi, \quad \phi^C = 1 \tag{4}$$

$$A^{CC} = A \tag{5}$$

$$A \cup A^C = 1, \quad A \cap A^C = \phi \tag{6}$$

SOLUTION:

From definition (2) we have

$$1^C = 1 - 1 = \phi \tag{7}$$

$$\phi^C = 1 - \phi = 1 \tag{8}$$

$$A^{CC} = (A^C)^C = (1 - A)^C =$$

$$= 1 - (1 - A) = \mathrm{A} \tag{9}$$

$$A \cup A^C = A \cup (1 - A) = 1 \tag{10}$$

$$A \cap A^C = A \cap (1 - A) = \phi \tag{11}$$

● PROBLEM 2-13

Prove DeMorgan's Theorem:

1. $(A \cup B)^C = A^C \cap B^C$ (1)

2. $(A \cap B)^C = A^C \cup B^C$ (2)

SOLUTION:

1.
$$[x \in (A \cup B)^C] \equiv [x \in 1 - (A \cup B)] \equiv$$

$$\equiv [x \in A \cup B]' \equiv [(x \in A) \vee (x \in B)]' \equiv$$

$$\equiv [(x \in A)' \wedge (x \in B)'] \equiv [(x \in A^C) \wedge (x \in B^C)] \equiv$$

$$\equiv [x \in A^C \cap B^C]. \tag{3}$$

Hence,

$$(A \cup B)^C = A^C \cap B^C. \tag{4}$$

2. Similarly

$$[x \in (A \cap B)^C] \equiv [x \in 1 - (A \cap B)] \equiv$$

$$\equiv [x \in A \cap B]' \equiv [(x \in A) \wedge (x \in B)]' \equiv$$

$$\equiv [(x \in A)' \vee (x \in B)'] \equiv [(x \in A^C) \vee (x \in B^C)] \equiv$$

$$\equiv [x \in A^C \cup B^C]. \tag{5}$$

Hence,

$$(A \cap B)^C = A^C \cup B^C. \tag{6}$$

DeMorgan's theorem can be generalized to any finite number of sets.

$$(A_1 \cup A_2 \cup \ldots \cup A_n)^C = A_1^{\ C} \cap A_2^{\ C} \cap \ldots \cap A_n^{\ C} \tag{7}$$

$$(A_1 \cap A_2 \cap \ldots \cap A_n)^C = A_1^{\ C} \cup A_2^{\ C} \cup \ldots \cup A_n^{\ C}. \tag{8}$$

● PROBLEM 2-14

Prove that:

1. $A - B = A \cap B^C$ (1)
2. $A \subset B$ if and only if $B^C \subset A^C$ (2)
3. $A^C - B^C = B - A$ (3)
4. Prove Equation (4) of Problem 2–11 using the notion of complement.

SOLUTION:

1.
$$(x \in A \cap B^C) \equiv [(x \in A) \wedge (x \in B^C)] \equiv$$

$$\equiv [(x \in A) \wedge (x \in 1 - B)] \equiv [(x \in A) \wedge (x \in B)'] \equiv$$

$$\equiv (x \in A - B). \tag{4}$$

2. Here, we shall apply the law of contraposition:

$$\text{if } \alpha \Rightarrow \beta, \text{ then } \beta' \Rightarrow \alpha' \tag{5}$$

$$(A \subset B) \equiv [(x \in A) \Rightarrow (x \in B)] \equiv$$

$$\equiv [(x \in B)' \Rightarrow (x \in A)'] \equiv [(x \notin B) \Rightarrow (x \notin A)] \equiv$$

$$\equiv [(x \in 1 - B) \Rightarrow (x \in 1 - A)] \equiv [(x \in B^C) \Rightarrow (x \in A^C)] \equiv$$

$$\equiv (B^C \subset A^C). \tag{6}$$

Thus,

$$A \subset B \text{ iff } B^C \subset A^C. \tag{7}$$

3. $$(x \in A^C - B^C) \equiv [(x \in A^C) \wedge (x \in B^C)'] \equiv$$

$$\equiv [(x \in 1 - A) \wedge (x \in 1 - B)'] \equiv [(x \in A)' \wedge (x \in B)''] \equiv$$

$$\equiv [(x \in B) \wedge (x \notin A)] \equiv (x \in B - A). \tag{8}$$

4. We have to prove that

$$(A \cap B) - (A \cap C) = A \cap (B - C) \tag{9}$$

By applying (1), we obtain

$$(A \cap B) - (A \cap C) = (A \cap B) \cap (A \cap C)^C =$$

$$= (A \cap B) \cap [1 - (A \cap C)] = (A \cap B) \cap (A^C \cup C^C) =$$

$$= (A \cap B \cap A^C) \cup (A \cap B \cap C^C) =$$

$$= (A \cap A^C \cap B) \cup (A \cap B \cap C^C) =$$

$$= A \cap B \cap C^C = A \cap (B - C). \tag{10}$$

In (10), we used DeMorgan's theorem.

• PROBLEM 2–15

Let $A_1, A_2, \ldots, A_n$ represent fixed subsets of the space 1. Let us denote

$$A_i^1 = 1 - A_i = A_i^C$$

$$A_i^0 = A_i. \tag{1}$$

A constituent of the space with respect to the sets $A_1, A_2, \ldots, A_n$ is any set of the form

$$A_1^{i_1} \cap A_2^{i_2} \cap \ldots \cap A_n^{i_n}$$

$$\text{where } i_1, \ldots, i_n = 1 \text{ or } 0. \tag{2}$$

1. Show that constituents are disjoint.

2. Prove that the union of all constituents is equal to 1.

SOLUTION:

1. Let $A_1^{i_1} \cap A_2^{i_2} \cap \ldots \cap A_n^{i_n}$ and $A_1^{k_1} \cap A_2^{k_2} \cap \ldots \cap A_n^{k_n}$ denote two constituents. Then at least one index i_l exists, such that

$$i_l \neq k_l. \tag{3}$$

Then the intersection of these two constituents is

$$A_1^{i_1} \cap A_2^{i_2} \ldots \cap A_n^{i_n} \cap A_1^{k_1} \cap A_2^{k_2} \cap \ldots \cap A_n^{k_n} \cap A_l^{i_l} \cap A_l^{k_l} =$$

$$= \ldots \cap A_l \cap A_l^C = \phi. \tag{4}$$

2. Now we shall show that the union of constituents is equal to 1.

Let $x \in 1$ represent an element of the space. We shall show that a constituent exists, such that x belongs to this constituent. Consider the sets $A_1, A_2, \ldots, A_n$. Two possibilities exist:

$$x \in A_1 \quad \text{or} \quad x \notin A_1.$$

If $x \in A_1 = A_1^0$, we set $i_1 = 0$. If $x \notin A_1$, then $x \in 1 - A_1 = A_1^1$ and we set $i_1 = 1$. We repeat the above procedure for each set $A_1, A_2, \ldots, A_n$ and obtain $i_1, i_2, \ldots, i_n$. The element x belongs to

$$\begin{gathered} x \in A_1^{i_1} \\ x \in A_2^{i_2} \\ \vdots \\ x \in A_n^{i_n} \end{gathered} \tag{5}$$

Hence,

$$x \in A_1^{i_1} \cap A_2^{i_2} \cap \ldots \cap A_n^{i_n} \tag{6}$$

Since x is an arbitrary element of 1, we conclude that the union of constituents is equal to 1.

Let A, B, and C represent the subsets of the space.

1. Find the constituents of the space with respect to the sets A, B, and C.

2. Represent the set

$$A-(B \cup C) \tag{1}$$

as the union of the constituents.

SOLUTION:

1. We shall consider all possible sets of the form

$$A^{i_1} \cap B^{i_2} \cap C^{i_3} \tag{2}$$

where

$$i_1, i_2, i_3 = 0 \text{ or } 1 \text{ and}$$

$$A^0 = A,\ B^0 = B,\ C^0 = C$$

$$A^1 = A^C,\ B^1 = B^C,\ C^1 = C^C. \tag{3}$$

The constituents are

$$A \cap B \cap C$$

$$A \cap B \cap C^C$$

$$A \cap B^C \cap C$$

$$A^C \cap B \cap C$$

$$A \cap B^C \cap C^C$$

$$A^C \cap B^C \cap C$$

$$A^C \cap B \cap C^C$$

$$A^C \cap B^C \cap C^C. \tag{4}$$

2. Let us write $A-(B \cup C)$ in the form

$$A-(B \cup C) = (A-B)-C = (A \cap B^C)-C =$$

$$= A \cap B^C \cap C^C. \tag{5}$$

Note that $A \cap B^C \cap C^C$ is itself a constituent.

● PROBLEM 2–17

The sets $A_1, A_2, \ldots, A_n$ are called independent if all the constituents are non-empty. Independent sets play an important role in the probability theory. Find the number of constituents of the independent sets $A_1, A_2, \ldots, A_n$.

SOLUTION:

The constituents are

$$A_1^{i_1} \cap A_2^{i_2} \cap \ldots \cap A_n^{i_n} \tag{1}$$

where $i_1, i_2, \ldots, i_n$ take one of two values, 0 or 1.

$$A_i^0 = A_i \text{ and } A_i^1 = 1 - A_i \tag{2}$$

The sets $A_1, A_2, \ldots, A_n$ are independent, therefore the constituents are non-empty sets. Setting

$$i_1 = i_2 = \ldots = i_n = 0 \tag{3}$$

in (1) we obtain

$$A_1 \cap A_2 \cap \ldots \cap A_n \neq \phi . \tag{4}$$

Thus, none of the sets $A_1, A_2, \ldots, A_n$ are empty sets. Similarly, setting $i_1 = i_2 = \ldots i_n = 1$, we find that

$$A_1^c \cap A_2^c \cap \ldots \cap A_n^c \neq \phi . \tag{5}$$

That is, none of the sets $A_1, A_2, \ldots, A_n$ are whole spaces.

Each of the superscripts $i_1, i_2, \ldots, i_n$ in

$$A_1^{i_1} \cap A_2^{i_2} \cap \ldots \cap A_n^{i_n}$$

takes up one of two values. Thus, the total number of combinations is

$$2 \times 2 \times \ \ldots \ \times 2 = 2^n. \tag{6}$$

The number of constituents is 2^n.

● PROBLEM 2–18

The symmetric difference of the sets A and B is defined by

$$A \dot{-} B = (A - B) \cup (B - A). \tag{1}$$

Prove the following formulas:

1. $A \dot{-} B = B \dot{-} A$ (2)

2. $A \dot{-} (B \dot{-} C) = (A \dot{-} B) \dot{-} C$ (3)
   The symmetric difference is associative.

3. $A \cup B = A \dot{-} B \dot{-} A \cap B$ (4)

SOLUTION:

We shall use

$$A - B = A \cap B^C \tag{5}$$

to write (1) in the form

$$A \dot{-} B = (A \cap B^C) \cup (A^C \cap B). \tag{6}$$

From (6) we obtain

$$(A \dot{-} B)^C = (A \cap B^C)^C \cap (A^C \cap B)^C =$$

$$= (A \cap B) \cup (A^C \cap B^C). \tag{7}$$

1. $$A \dot{-} B = (A - B) \cup (B - A) = (B - A) \cup (A - B) =$$

$$= B \dot{-} A. \tag{8}$$

2. $$A \dot{-} (B \dot{-} C) = [A \cap (B \dot{-} C)^C] \cup [(B \dot{-} C) \cap A^C] = \tag{9}$$

$$=\{A \cap [(C \cap B) \cup (C^C \cap B^C)]\} \cup \{[(B \cap C^C) \cup (B^C \cap C)] \cap A^C\} =$$

$$= (A \cap B \cap C) \cup (A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C) \cup (A^C \cap B^C \cap C).$$

Similarly,

$$(A \dot{-} B) \dot{-} C = [(A \cap B^C) \cup (A^C \cap B)] \dot{-} C = \tag{10}$$

$$= \{[(A \cap B^C) \cup (A^C \cap B)] \cap C^C\} \cup [(A \dot{-} B)^C \cap C] =$$

$$= (A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C) \cup \{[(A \cap B) \cup (A^C \cap B^C)] \cap C\} =$$

$$= (A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C) \cup (A \cap B \cap C) \cup (A^C \cap B^C \cap C).$$

Comparing (9) and (10) we find

$$(A \dotdiv B) \dotdiv C = A \dotdiv (B \dotdiv C) = A \dotdiv B \dotdiv C. \tag{11}$$

3.
$$A \dotdiv B \dotdiv A \cap B = (A \dotdiv B) \dotdiv (A \cap B) =$$
$$= [(A \cap B^C) \cup (A^C \cap B)] \dotdiv (A \cap B) =$$
$$= \{[(A \cap B^C) \cup (A^C \cap B)] \cap (A^C \cup B^C)\} \cup$$
$$\cup \{[(A^C \cup B) \cap (A \cup B^C)] \cap (A \cap B)\} =$$
$$= [(A \cap B^C) \cap (A^C \cup B^C)] \cup [(A^C \cap B) \cap (A^C \cup B^C)] \cup$$
$$\cup [(A^C \cap B^C) \cap (A \cap B)] \cup [(A \cap B) \cap (A \cap B)] =$$
$$= (A \cap B^C) \cup (A^C \cap B) \cup (A \cap B) =$$
$$= (A - B) \cup (B - A) \cup (A \cap B) = A \cup B. \tag{12}$$

Similarly, we can show that

$$A - B = A \dotdiv A \cap B. \tag{13}$$

● PROBLEM 2-19

Show that

$$(A_1 \cup \ldots \cup A_n) \dotdiv (B_1 \cup \ldots \cup B_n) \subset (A_1 \dotdiv B_1) \cup \ldots \cup (A_n \dotdiv B_n) \tag{1}$$

where

$$A \dotdiv B = (A - B) \cup (B - A). \tag{2}$$

SOLUTION:

Let us denote

$$A = A_1 \cup \ldots \cup A_n$$
$$B = B_1 \cup \ldots \cup B_n. \tag{3}$$

Suppose

$$x \in A \dotdiv B = (A - B) \cup (B - A). \tag{4}$$

The sets $A - B$ and $B - A$ are disjoint sets (i.e., they have no common elements). We can assume that

$$x \in A - B, \tag{5}$$

then $x \in A$ and $(x \notin B) \equiv (x \in B)'$. We have

$$x \in A_1 \cup \ldots \cup A_n \tag{6}$$

and

$$x \notin B_1 \cup \ldots \cup B_n. \tag{7}$$

Therefore a set A_l exists, such that

$$x \in A_l \tag{8}$$

and

$$\begin{aligned}(x \in B)' &\equiv (x \in B_1 \cup \ldots \cup B_n)' \equiv \\ &\equiv [(x \in B_1) \vee (x \in B_2) \vee \ldots \vee (x \in B_n)]' \equiv \\ &\equiv (x \in B_1)' \wedge (x \in B_2)' \wedge \ldots \wedge (x \in B_n)' \equiv \\ &\equiv (x \notin B_1) \wedge (x \notin B_2) \wedge \ldots \wedge (x \notin B_n).\end{aligned} \tag{9}$$

Thus,

$$x \in A_l \text{ and } x \notin B_l \tag{10}$$

and

$$x \in A_l \dotminus B_l = (A_l - B_l) \cup (B_l - A_l). \tag{11}$$

Hence,

$$(A_1 \cap \ldots \cap A_n) \dotminus (B_1 \cap \ldots \cap B_n) \subset (A_1 \dotminus B_1) \cup \ldots \cup (A_n \dotminus B_n). \tag{12}$$

● PROBLEM 2-20

DEFINITION OF A RING

The Operations + and · form a commutative ring, if they satisfy the conditions:

1. $x + y = y + x\,,\ x \cdot y = y \cdot x$ (1)

2. $x + (y + z) = (x + y) + z,\ x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (2)

3. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ (3)

4. An element 0 exists, such that

$$x + 0 = x \tag{4}$$

5. For every pair x, y, an element z exists, such that

$$x = y + z. \tag{5}$$

■

Show that the sets form a ring with respect to the operations $\dot{-}$ and $\cap$ but don't form a ring with respect to the operations $\cup$ and $\cap$.

SOLUTION:

Let us replace in (1) - (5) operation + by $\dot{-}$, and operation $\cdot$ by $\cap$. We obtain

$$A \dot{-} B = B \dot{-} A, \quad A \cap B = B \cap A \tag{6}$$

which are obviously true. Then

$$A \dot{-} (B \dot{-} C) = (A \dot{-} B) \dot{-} C, \quad A \cap (B \cap C) = (A \cap B) \cap C. \tag{7}$$

The relationship

$$A \cap (B \dot{-} C) = (A \cap B) \dot{-} (A \cap C) \tag{8}$$

is true.

For an empty set ϕ, we obtain

$$A \dot{-} \phi = A. \tag{9}$$

To prove condition 5, we have to show that

$$(A = B \dot{-} C) \Rightarrow (C = B \dot{-} A) \tag{10}$$

which is easy to prove because

$$A = B \dot{-} C = B \dot{-} B \dot{-} A = (B \dot{-} B) \dot{-} A = A. \tag{11}$$

Hence, operations $\dot{-}$ and $\cap$ form a ring.

Consider operations $\cup$ and $\cap$. We have

$$A \cup B = B \cup A, \quad A \cap B = B \cap A \tag{12}$$

$$A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap= B) \cap C. \tag{13}$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{14}$$

$$A \cup \{\phi\} = A \tag{15}$$

It is not true that for every pair A, B, a set C exists, such that

$$A = B \cup C. \tag{16}$$

For example, let $A = \{\phi\}$ and $C \neq \{\phi\}$. Thus, $\cup$ and $\cap$ do not form a ring.

● PROBLEM 2-21

1. Show that the family of all subsets of a given set A is an ideal.

2. Show that the family of all sets B, such that $A \subset B \subset 1$, is a filter.

SOLUTION:

DEFINITION OF IDEAL

A family is a collection of sets. A non-empty family I of subsets of 1 is called ideal if

$$(A \in I) \wedge (B \subset A) \Rightarrow B \in I \tag{1}$$

$$(A \in I) \wedge (B \in I) \Rightarrow (A \cup B \in I). \tag{2}$$

1. Let P denote the family of all subsets of a given set A. If $D \in P$, then $D \subset A$. If $G \subset D$, then $G \subset A$ and $G \in P$. Hence, condition (1) is fulfilled.

If $D \in P$ and $G \in P$, then $D \subset A$ and $G \subset A$. Hence, $D \cup G \subset A$ and $D \cup G \in P$. Therefore, the family of all subsets of a given set is an ideal.

DEFINITION OF A FILTER

A non-empty family F is called a filter if

$$(A \in F) \wedge (A \subset B) \Rightarrow (B \in F) \tag{3}$$

$$(A \in F) \wedge (B \in F) \Rightarrow (A \cap B \in F) \tag{4}$$

■

2. Let A represent a given set and R the family of all sets B, such that $A \subset B \subset 1$. Suppose $D \in R$ and $D \subset G$, then $A \subset D \subset G \subset 1$ and $G \in R$. Condition (3) is fulfilled.

If $D \in R$ and $G \in R$, then $A \subset D$ and $A \subset G$. Therefore, $A \subset D \cap G$ and $D \cap G \in R$. Hence, R is a filter.

● PROBLEM 2-22

Show that a family of sets is a filter if and only if the family of the complements of these sets is an ideal.

SOLUTION:

Let $S = \{A, B, C, \ldots\}$ represent a family of sets. We shall prove that

$$\begin{pmatrix} S = \{A,B,C,\ldots\} \text{ is} \\ \text{a filter} \end{pmatrix} \Rightarrow \begin{pmatrix} P = \{A^c,B^c,C^c,\ldots\} \text{ is} \\ \text{an ideal} \end{pmatrix}$$

Let $A^C \in P$ and $B^C \subset A^C$, then $A \subset B$. But $A \in S$ (which is a filter); hence, $B \in S$. Therefore $B^C \in P$.

Let $A^C \in P$ and $B^C \in P$, then $A \in S$ and $B \in S$. Since S is a filter, $A \cap B \in S$.

But $A^C \cup B^C = (A \cap B)^C \in P$.

Similarly, we can show that

$$\begin{pmatrix} P = \{A^c,B^c,\ldots\} \text{ is} \\ \text{an ideal} \end{pmatrix} \Rightarrow \begin{pmatrix} S = \{A,B,C,\ldots\} \text{ is} \\ \text{a filter} \end{pmatrix}$$

Let $A \in S$ and $A \subset B$. Then $A^C \in P$ and $B^C \subset A^C$. Hence, $B^C \in P$ and $B \in S$.

Let $A \in S$ and $B \in S$. Then $A^C \in P$ and $B^C \in P$.

Since P is an ideal, $A^C \cup B^C \in P$ and $A \cap B$ $(A^C \cup B^C)^C \in S$.

● PROBLEM 2-23

In 1895, Georg Cantor created a theory of sets. At that time, it was accepted that a universal set (that is, "the set of all sets") exists. In 1902, Bertrand Russell showed in his famous paradox that the admission of a set of all sets leads to a contradiction. Explain Russell's paradox.

SOLUTION:

We shall show that the assumption that the universal set exists leads to contradictory statements.

THEOREM 1

Suppose that the set of all sets, denoted by W, exists. Let

$$A = \{x \in W : x \notin x\}$$

Then

or

$A \notin A$ $\qquad$ $A \in A$

Indeed,

$$(x \in A) \equiv (x \notin x); \text{ therefore } (A \in A) \equiv (A \notin A).$$

That leads to:

THEOREM 2

A set of all sets does not exist.

This fact was briefly described by Paul Halmos in this statement: "Nothing contains everything." New paradoxes of set theory appeared shortly after Russell's paradox. To overcome this difficulty, mathematicians created several variations of axiomatic set theory. None of these theories was completely satisfactory, nevertheless they eliminated most of the antinomies.

● PROBLEM 2–24

Consider the axiomatic formulation of the algebra of sets. It consists of the concept of element, of set and of the relation of an element belonging to a set ($x \in A$) and four fundamental axioms. Using this system, show that

1. the null set exists.

2. for any two sets A and B, only one set $A \cup B$ exists.

SOLUTION:

We shall list the four fundamental axioms.

I. UNIQUENESS AXIOM (Axiom of Extension).

If the sets A and B have the same elements, then A and B are identical.

II. UNION AXIOM.

For two arbitrary sets A and B, a set exists which contains all the elements of the set A, and all the elements of the set B and which does not contain any other element.

III. DIFFERENCE AXIOM.

For two arbitrary sets A and B, a set exists which contains those and only those elements of the set A which are not elements of the set B.

IV. EXISTENCE AXIOM.

At least one set exists.

1. Let us define the null set by the formula

$$\phi = A - A.$$

The existence of at least one set is guaranteed by Axiom IV.

2. For given sets A and B, one and only one set satisfying Axiom II exists. Hence, the operation $\cup$ is unique.

Since

$$A \cap B = A - (A - B),$$

it is not necessary to include an axiom on the existence of an intersection. The intersection can be defined in terms of the difference.

● PROBLEM 2-25

Explain the system of axioms called Boolean algebra.

SOLUTION:

Note that in most of the theorems of the algebra of sets the symbol $\in$ does not appear, although it occurs in their proofs. This suggests the possibility of establishing the system of axioms, which will enable us to prove the theorems without referring to the relation $\in$.

BOOLEAN ALGEBRA

The fundamental concepts are:

The set ϕ and the operations $\cup$, $\cap$, $-$.

We assume the following axioms:

1. $A \cup B = B \cup A$

2. $A \cap B = B \cap A$

3. $A \cup (B \cup C) = (A \cup B) \cup C$

4. $A \cap (B \cap C) = (A \cap B) \cap C$

5. $A \cup \phi = A$

6. $A \cap (A \cup B) = A$

7. $A \cup (A \cap B) = A$

8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

9. $(A - B) \cup B = A \cup B$

10. $B \cap (A - B) = \phi$

11. $A \cap 1 = A$

12. $(A \subset B) \equiv (A \cup B = B)$

From the above axioms we can deduce all the theorems of the algebra of sets. Axiom 12 defines the Inclusion Theory of sets as only one of the applications of Boolean algebra. Replacing in axioms the notion of a set by sentences, we obtain the algebra of sentences. This explains the close relationship between the algebra of sentences and the algebra of sets.

Sentence		Sets
$\vee$	corresponds to	$\cup$
$\wedge$	corresponds to	$\cap$
$'$ (negation)	corresponds to	^{c} (complement)

By omitting axioms 8 - 12, we obtain the axiomatic formulation of the lattice theory, applied, for example, in electrical networks.

CHAPTER 3

EQUIVALENCE RELATIONS, AXIOMATIC SET THEORY, QUANTIFIERS

1. The set of all values of the variable x, for which $\varphi(x)$ is a true sentence, is denoted by the symbol

$$\{x : \varphi(x)\}. \tag{1}$$

The following equivalence holds:

$$\text{For every } a\text{: } [a \in \{x : \varphi(x)\}] \equiv \varphi(a). \tag{2}$$

Using (1) and (2), describe the set of the real numbers which are larger than zero and smaller than or equal to 5.

2. Prove the following formulas:

$$\{x : \varphi(x) \vee \zeta(x)\} = \{x : \varphi(x)\} \cup \{x : \zeta(x)\} \tag{3}$$

$$\{x : \zeta(x) \wedge \varphi(x)\} = \{x : \zeta(x)\} \cap \{x : \varphi(x)\} \tag{4}$$

$$\{x : \varphi(x) \wedge [\zeta(x)]'\} = \{x : \varphi(x)\} - \{x : \zeta(x)\}$$

$$\text{and } \{x : [\varphi(x)]'\} = \{x : \varphi(x)\}^C. \tag{5}$$

SOLUTION:

1. The set can be described as follows:

$$\{x : (x \in R) \wedge (x > 0) \wedge (x \leq 5)\}. \tag{6}$$

2.

$$a \in \{x : \varphi(x) \vee \zeta(x)\} \equiv \varphi(a) \vee \zeta(a) \equiv$$

$$\equiv [a \in \{x : \varphi(x)\}] \vee [a \in \{x : \zeta(x)\}] \equiv$$

$$\equiv a \in \{x : \varphi(x)\} \cup \{x : \zeta(x)\}. \tag{7}$$

We employ the strategy used in the solution of (3) to prove (4).

$$a \in \{x : \zeta(x) \wedge \varphi(x)\} \equiv \zeta(a) \wedge \varphi(a) \equiv$$

$$\equiv [a \in \{x : \zeta(x)\}] \wedge [a \in \{x : \varphi(x)\}] \equiv$$

$$\equiv a \in \{x : \zeta(x)\} \cap \{x : \varphi(x)\}. \tag{8}$$

Proving (5),

$$a \in \{x : \varphi(x) \wedge [\zeta(x)]'\} \equiv \varphi(a) \wedge [\zeta(a)]' \equiv$$

$$\equiv [a \in \{x : \varphi(x)\}] \wedge [a \notin \{x : \zeta(x)\}] \equiv$$

$$\equiv a \in \{x : \varphi(x)\} - \{x : \zeta(x)\}. \tag{9}$$

Proving (6),

$$a \in \{x : [\varphi(x)]'\} \equiv [\varphi(a)]' \equiv$$

$$\equiv (a \in 1) \wedge [a \notin \{x : \varphi(x)\}] \equiv$$

$$\equiv a \in \{x : \varphi(x)\}^C. \tag{10}$$

The symbol $\{x : \varphi(x)\}$ will be used very often, sometimes with some modifications, throughout this book.

● PROBLEM 3-2

1. Using qualifiers, write the sentence: For every natural number, a larger natural number exists.

2. Explain the quantifiers by using the calculus of sentences.

SOLUTION:

1. The quantifier "there exists" is denoted by $\exists$ and the quantifier "for each" is denoted by $\forall$.

Let A represent the domain of a general statement $p(x)$, i.e., $x \in A$. Then

$$\forall x \in A : p(x) \tag{1}$$

asserts that for all $x \in A$, the statement $p(x)$ is true. Similarly,

$$\exists x \in A : p(x) \tag{2}$$

means that at least one $x \in A$ exists, such that $p(x)$ is true.

By denoting the set of natural numbers by N, we write

$$\forall n \, \exists m : n < m \tag{3}$$

where $m, n \in N$.

2. Suppose the elements of the set A can be written in the form $a_1, a_2, a_3, \ldots$. The sentence $\forall x : p(x)$ means that $p(x)$ is true for each $a_1, a_2, a_3, \ldots$. Thus

$$\forall x : p(x) \text{ is equivalent to } p(a_1) \wedge p(a_2) \wedge \ldots \tag{4}$$

Similarly, the sentence $\exists x : p(x)$ means that $p(x)$ is true for at least one of

the elements $a_1, a_2, a_3, \ldots$. Thus

$$\exists x : p(x) \text{ is equivalent to } p(a_1) \vee p(a_2) \vee p(a_3) \ldots \tag{5}$$

Note that a universal quantifier $\forall$ is "stronger" than existential quantifier $\exists$.

$$[\forall x : p(x)] \Rightarrow [\exists x : p(x)]. \tag{6}$$

● PROBLEM 3-3

1. Using quantifiers write the statement: For each x there exists a y such that for all z the statement $p(x, y, z)$ is true.

2. Write the negation of this statement.

SOLUTION:

We shall apply the following rule of quantifier negation:

$$[\forall x : p(x)]' \equiv \exists x : [p(x)]' \tag{1}$$

$$[\exists x : p(x)]' \equiv \forall x : [p(x)]' \tag{2}$$

Equations (1) and (2) are generalized DeMorgan formulas.

Note that by taking negation of (1), we can define a universal quantifier in terms of the existential quantifier and negation

$$\forall x : p(x) \equiv [\exists x : [p(x)]']'. \tag{3}$$

Similarly, existential quantifiers can be defined in terms of the universal quantifier and negation

$$\exists x : p(x) \equiv [\forall x : [p(x)]']'. \tag{4}$$

1. This statement can be written as

$$\forall x \, \exists y \, \forall z : p(x, y, z). \tag{5}$$

2. The negation of (5) is

$$[\forall x \, \exists y \, \forall z : p(x, y, z)]' \equiv$$

$$\equiv [\forall x \, [\exists y \, \forall z : p(x, y, z)]]' \equiv$$

$$\equiv \exists x \, [\exists y \, \forall z : p(x, y, z)]'. \tag{6}$$

Here we used (1).

$$\equiv \exists x\, [\exists y\, [\forall z : p(x, y, z)]]' \equiv$$

$$\equiv \exists x\, \forall y\, [\forall z : p(x, y, z)]' \equiv$$

$$\equiv \exists x\, \forall y\, \exists z : [p(x, y, z)]' \equiv \tag{7}$$

● PROBLEM 3-4

Using quantifiers, write down the definitions of:

1. Limit of a sequence of real numbers.
2. Continuous functions.
3. Uniformly continuous functions.

SOLUTION:

1. Let (x_n) denote a sequence of real numbers. This sequence is said to have a limit x. We write

$$\lim_{n \to \infty} x_n = x \tag{1}$$

when

$$\forall \varepsilon > 0 \quad \exists m \in N \quad \forall n \geq m \quad |x_n - x| < \varepsilon. \tag{2}$$

2. Let $f : R^1 \to R^1$ be a function continuous at x_0. Then

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in R^1 \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \tag{3}$$

Function $f(x)$ is continuous at x_0 if, for every $\varepsilon > 0$, $\delta > 0$ exists such that for every $x \in R^1$, if $|x - x_0| < \delta$, implies $|f(x) - f(x_0)| < \varepsilon$.

3. Function $f : R^1 \to R^1$ is said to be uniformly continuous on R^1 if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, x' \in R^1 \quad |x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon. \tag{4}$$

From (3) and (4) we conclude that if a function is uniformly continuous on R^1, then it is continuous on R^1. The opposite is not true. For example, function $f(x) = x^2$ is continuous on R^1 but not uniformly continuous.

1. Is the following definition of an ordered pair correct?

$$(a, b) = \{\{a\}, \{a, b\}\}. \tag{1}$$

2. Find the Cartesian product $A \times B$, where $A = [1, 2]$ and $B = [2, 3]$.

3. Let Z denote the set of all positive and negative integers and zero. Find $A \times B$, where $A = B = Z$.

SOLUTION:

1. To each of two elements a, b there corresponds their ordered pair (a, b), which satisfies the condition

$$(a, b) = (c, d) \text{ if, and only if } a = c \text{ and } b = d. \tag{2}$$

Hence, in general $(a, b) \neq (b, a)$. But

$$(a, b) = (b, a) \text{ iff } a = b. \tag{3}$$

Definition (1) satisfies condition (2) because

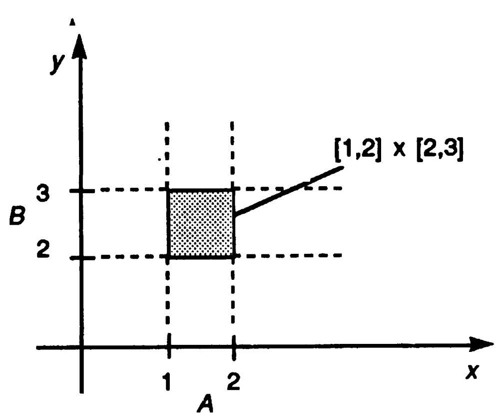
$$\{a, \{a, b\}\} = \{c, \{c, d\}\} \text{ if, and only if } a = c \text{ and } b = d.$$

Thus, definition (1) is correct.

2. **DEFINITION**

Let A and B represent two sets. Their Cartesian product, $A \times B$, is the set of all ordered couples

$$A \times B = \{(a, b) : a \in A,\ b \in B\}. \tag{4}$$



The Cartesian product is represented by the shaded area.

FIGURE 1.

3. The set $A \times B$, where $A = B = Z$ represents the set of lattice points in R^2.

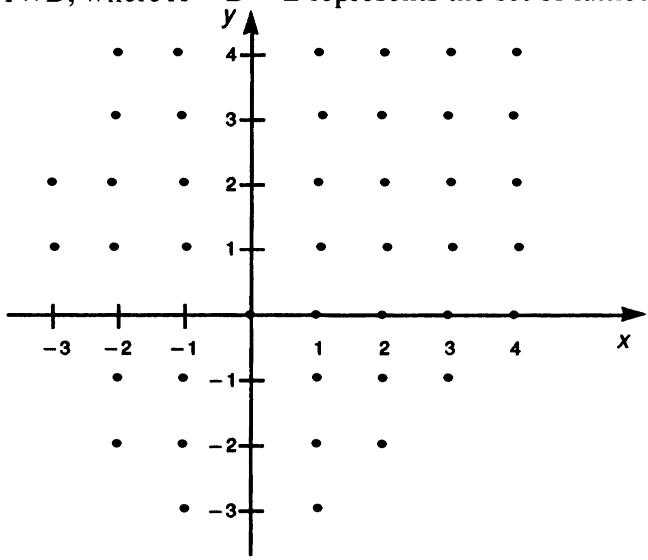


FIGURE 2.

Dots represent the elements of the set $Z \times Z$.

● PROBLEM 3-6

1. Let A represent the circle in the xy-plane, $x^2 + y^2 \le 1$ and let B represent the set of points along the z coordinate, such that $0 \le z \le 1$. Find $A \times B$.

2. Show that,

$$(A \times B = \phi) \Leftrightarrow [(A = \phi) \vee (B = \phi)]. \tag{1}$$

3. Show that,

$$\text{if } A \times B \ne \phi, \text{ then } A \times B \subset C \times D \text{ iff}$$

$$(A \subset C) \wedge (B \subset D). \tag{2}$$

SOLUTION:

1. The set $A \times B$ is a cylinder of altitude 1, shown in Figure 1.

2. If $A \times B \ne \phi$, then an element exists $(a, b) \in A \times B$. Thus, $a \in A$ and $b \in B$ and $A \ne \phi$ and $B \ne \phi$.

Similarly, if $A \neq \phi$ and $B \neq \phi$, then elements a, b exist, such that $a \in A$ and $b \in B$. Therefore, $(a, b) \in A \times B$ and $A \times B \neq \phi$.

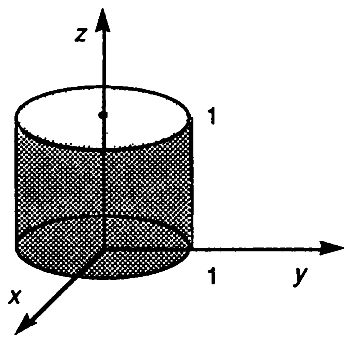


FIGURE 1.

3. We shall prove that

$$(A \times B \subset C \times D) \Leftrightarrow (A \subset C) \wedge (B \subset D). \tag{3}$$

$\Rightarrow$

Let $(a, b) \in A \times B$. Then $(a, b) \in C \times D$. Therefore, if $a \in A$, then $a \in C$. Similarly, if $b \in B$, then $b \in D$.

$\Leftarrow$

Let $(a, b) \in A \times B$. Then $a \in A$ and $a \in C$. Also $b \in B$ and $b \in D$. Therefore $(a, b) \in C \times D$ and

$$A \times B \subset C \times D. \tag{4}$$

● PROBLEM 3-7

Show that

$$A \times (B \cup C) = A \times B \cup A \times C \tag{1}$$

$$A \times (B \cap C) = A \times B \cap A \times C \tag{2}$$

$$A \times (B - C) = A \times B - A \times C \tag{3}$$

SOLUTION:

$$A \times (B \cup C) = \{(x, y) : (x \in A) \wedge (y \in B \cup C)\} =$$

$$= \{(x, y) : (x \in A) \wedge [(y \in B) \vee (y \in C)]\} =$$

$$= \{(x, y) : [(x \in A) \wedge (y \in B)] \vee [(x \in A) \wedge (y \in C)]\} =$$

$$= \{(x, y) : [(x, y) \in A \times B] \vee [(x, y) \in A \times C]\} =$$

$$= A \times B \cup A \times C. \tag{4}$$

Similarly, we prove (2).

$$A \times (B \cap C) = \{(x, y) : (x \in A) \wedge (y \in B \cap C)\} =$$

$$= \{(x, y) : (x \in A) \wedge [(y \in B) \wedge (y \in C)]\} =$$

$$= \{(x, y) : [(x, y) \in A \times B] \wedge [(x, y) \in A \times C]\} =$$

$$= A \times B \cap A \times C. \tag{5}$$

And (3).

$$A \times (B - C) = \{(x, y) : (x \in A) \wedge (y \in B - C)\} =$$

$$= \{(x, y) : (x \in A) \wedge [(y \in B) \wedge (y \notin C)]\} =$$

$$= \{(x, y) : [(x, y) \in A \times B] \wedge [(x, y) \notin A \times C]\} =$$

$$= A \times B - A \times C. \tag{6}$$

• PROBLEM 3-8

1. Prove that:

$$(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D) \tag{1}$$

$$(A \times C) \cup (B \times D) \subset (A \cup B) \times (C \cup D) \tag{2}$$

where $A, B \subset X$ and $C, D \subset Y$.

2.
$$A \times B = (A \times Y) \cap (X \times B) \tag{3}$$

$$(A \times B)^C = (A^C \times Y) \cup (X \times B^C) \tag{4}$$

where $A \subset X$ and $B \subset Y$. The symbols A^c and B^c denote the complements with respect to X and Y, and $(A \times B)^c$ denotes the complement with respect to $X \times Y$.

SOLUTION:

1.
$$[(x, y) \in (A \times C) \cap (B \times D)] \equiv$$

$$\equiv [(x, y) \in A \times C] \wedge [(x, y) \in B \times D] \equiv$$

$$\equiv (x \in A) \wedge (y \in C) \wedge (x \in B) \wedge (y \in D) \equiv$$

$$\equiv [(x \in A) \wedge (x \in B)] \wedge [(y \in C) \wedge (y \in D)] \equiv$$

$$\equiv (x \in A \cap B) \wedge (y \in C \cap D) \equiv$$

$$\equiv (x, y) \in (A \cap B) \times (C \cap D). \tag{5}$$

Applying Equation (1) of Problem 3-7 we find:

$$(A \cup B) \times (C \cup D) = [A \times (C \cup D)] \cup [B \times (C \cup D)] =$$

$$= (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D). \tag{6}$$

Equation (2) follows from (6):

$$(A \times C) \cup (B \times D) \subset (A \cup B) \times (C \cup D). \tag{7}$$

2. Let us write (1) in the form:

$$(A \times Y) \cap (X \times B) = (A \cap X) \times (Y \cap B). \tag{8}$$

Since $A \subset X$, $A \cap X = A$. Similarly, since $B \subset Y$, $Y \cap B = B$.
Therefore,

$$A \times B = (A \times Y) \cap (X \times B) . \tag{9}$$

To prove (4), we shall apply DeMorgan's law. Since

$$A \times B = (A \cap X) \times (Y \cap B) = (A \times Y) \cap (X \times B) \tag{10}$$

we obtain

$$(A \times B)^c = (A \times Y)^c \cup (X \times B)^c. \tag{11}$$

But

$$(A \times Y)^c = A^c \times Y \quad \text{and} \tag{12}$$

$$(X \times B)^c = X \times B^c. \tag{13}$$

Therefore

$$(A \times B)^C = (A^C \times Y) \cup (X \times B^C). \tag{14}$$

● PROBLEM 3-9

1. Give a definition of a family of sets and show that

$$\{A_n : n \in N\} \tag{1}$$

where $A_n =]-n, n[$ is a family of sets.

2. Is it true, that any set of sets can be converted to a family of sets?

SOLUTION:

DEFINITION OF A FAMILY OF SETS

The collection of sets

$$\{A_\lambda : \lambda \in \Lambda\} \tag{2}$$

is called a family of sets if, to each element λ of some set $\Lambda \neq \phi$, a set A_λ corresponds. Set Λ is called an indexing set for the family.

Unlike the definition of a set, where all elements have to be different, in a family of sets it is not required that sets with distinct indices be different. Equation (1) defines a family of sets, where N is the indexing set. Set A_n is an open interval

$$]-n, n[= \{x \in R' : -n < x < n\}. \tag{3}$$

2. Let G represent any set of sets. Then we can use G itself as an indexing set and assign the set it represents to each member of the set G.

● PROBLEM 3-10

Let

$$A_n =]-\frac{1}{n}, \frac{1}{n}[\tag{1}$$

where the indexing set is the set of all positive integers Z^+

$$Z^+ = \{1, 2, 3, \dots\}. \tag{2}$$

Find

$$\bigcup_n A_n$$

and

$$\bigcap_n A_n .$$

SOLUTION:

Let X represent a given set, and $\{A_\lambda : \lambda \in \Lambda\}$ a family of subsets of X. The union, denoted by $\bigcup\limits_\lambda A_\lambda$, is the set

$$\{x \in X : \exists \lambda \in \Lambda; x \in A_\lambda\}. \tag{3}$$

Hence, for the family of sets defined by (1), we obtain

$$\bigcup_n A_n = \bigcup_n \left] -\frac{1}{n}, \frac{1}{n} \right[=]-1,1[. \tag{4}$$

The intersection, denoted by $\bigcap\limits_\lambda A_\lambda$, is the set

$$\{x \in X : \forall \lambda \in \Lambda; x \in A_\lambda\}. \tag{5}$$

Thus

$$\bigcap_n A_n = \bigcap_n \left] -\frac{1}{n}, \frac{1}{n} \right[= \{0\} \tag{6}$$

since 0 is the only common point of all intervals $\left] -\frac{1}{n}, \frac{1}{n} \right[$.

Obviously, the union and intersection of a family of sets does not depend on how the family is indexed.

● PROBLEM 3-11

1. Show that $\bigcap\limits_\lambda$ is distributive over $\cup$ and $\bigcup\limits_\lambda$ is distributive over $\cap$. That is, show that:

$$[\cap \{A_\lambda : \lambda \in \Lambda\}] \cup [\cap \{B_\omega : \omega \in \Omega\}] =$$

$$= \cap \{A_\lambda \cup B_\omega : (\lambda, \omega) \in \Lambda \times \Omega\} \tag{1}$$

$$[\cup \{A_\lambda : \lambda \in \Lambda\}] \cap [\cup \{B_\omega : \omega \in \Omega\}] =$$

$$= \cup \{A_\lambda \cap B_\omega : (\lambda, \omega) \in \Lambda \times \Omega\} \tag{2}$$

2. Show that

$$\left(\bigcup_{\lambda} A_{\lambda}\right)^{C} = \bigcap_{\lambda} A_{\lambda}^{C} \tag{3}$$

and

$$\left(\bigcap_{\lambda} A_{\lambda}\right)^{C} = \bigcup_{\lambda} A_{\lambda}^{C} \tag{4}$$

where complements are taken with respect to X.

SOLUTION:

1.
$$x \in [\cap \{A_\lambda : \lambda \in \Lambda\}] \cup [\cap \{B_\omega : \omega \in \Omega\}] \equiv$$

$$\equiv [\forall \lambda \in \Lambda; x \in A_\lambda] \vee [\forall \omega \in \Omega; x \in B_\omega] \equiv$$

$$\equiv [\forall (\lambda, \omega) \in \Lambda \times \Omega : x \in A_\lambda \cup B_\omega] \equiv$$

$$\equiv x \in \cap \{A_\lambda \cup B_\omega : (\lambda, \omega) \in \Lambda \times \Omega\}. \tag{5}$$

In the same manner, we prove (2)

$$x \in [\cup \{A_\lambda : \lambda \in \Lambda\}] \cap [\cup \{B_\omega : \omega \in \Omega\}] \equiv$$

$$\equiv [x \in \cup A_\lambda] \wedge [x \in \cup B_\omega] \equiv$$

$$\equiv (\exists \lambda \in \Lambda : x \in A_\lambda) \wedge (\exists \omega \in \Omega : x \in B_\omega)$$

$$\equiv [\exists (\lambda, \omega) \in \Lambda \times \Omega : x \in A_\lambda \cap B_\omega] \equiv$$

$$\equiv x \in \cup \{A_\lambda \cap B_\omega : (\lambda, \omega) \in \Lambda \times \Omega\}. \tag{6}$$

2.
$$x \in \left(\bigcup_{\lambda} A_\lambda\right)^{C} \equiv x \in X - \left(\bigcup_{\lambda} A_\lambda\right) \equiv$$

$$\equiv x \notin \left(\bigcup_{\lambda} A_\lambda\right) \equiv [\exists \lambda \in \Lambda : x \in A_\lambda]' \equiv$$

$$\equiv \forall \lambda \in \Lambda : x \notin A_\lambda \equiv \forall \lambda \in \Lambda : x \in A_\lambda^{C} \equiv$$

$$\equiv x \in \bigcap_{\lambda} A_\lambda^{C}. \tag{7}$$

Similarly,
$$x \in \left(\bigcap_{\lambda} A_\lambda\right)^{C} \equiv x \notin \left(\bigcap_{\lambda} A_\lambda\right) \equiv$$

$$\equiv [\forall \lambda \in \Lambda : x \in A_\lambda]' \equiv [\exists \lambda \in \Lambda : x \notin A_\lambda] \equiv$$

$$\equiv [\exists \lambda \in \Lambda : x \in A_\lambda^{C}] \equiv x \in \bigcup_{\lambda} A_\lambda^{C}. \tag{8}$$

Let $\{A_n : n \in N\}$ be a family of sets and let

$$B_k = A_0 \cup A_1 \cup \ldots \cup A_k, \quad k = 0, 1, 2, \ldots \tag{1}$$

Show that the union

$$\bigcup_0^\infty A_n = A_0 \cup (A_1 - B_0) \cup \ldots \cup (A_n - B_{n-1}) \cup \ldots \tag{2}$$

is pairwise disjoint.

SOLUTION:

First, we shall prove (2). Suppose

$$x \in \bigcup_0^\infty A_n \tag{3}$$

and $x \in A_0$. Then $x \in A_0 \cup (A_1 - B_0) \cup \ldots$. Now, suppose A_m is the first set in the sequence

$$A_0, A_1, \ldots, A_m, \ldots \tag{4}$$

such that

$$x \notin A_0, x \notin A_1, \ldots, x \notin A_{m-1}, x \in A_m, \ldots \tag{5}$$

Then

$$x \in \bigcup_0^\infty A_n \text{ and}$$

$$x \in A_m - B_{m-1} = A_m - (A_0 \cup \ldots \cup A_{m-1}). \tag{6}$$

That proves (2).

To show that the union

$$A_0 \cup (A_1 - B_0) \cup \ldots \cup (A_m - B_{m-1}) \cup \ldots \tag{7}$$

consists of pairwise disjoint sets, suppose

$$x \notin A_0, x \notin A_1, \ldots, x \notin A_{m-1}, x \in A_m, \tag{8}$$

Then

$$x \notin A_0, x \notin A_1 - B_0, x \notin A_2 - B_1, \ldots \tag{9}$$

but

$$x \in A_m - B_{m-1}. \tag{10}$$

Again

$$x \notin A_{m+1} - B_m = A_{m+1} - (A_1 \cup \ldots \cup A_m). \tag{11}$$

Hence the sets $A_0, A_1 - B_0, \ldots, A_n - B_{n-1}, \ldots$ in (7) are pairwise disjoint.

1. Let ρ denote the relation < on $A \times B$, where $A = \{1, 3, 4, 5\}$ and $B = \{3, 4, 5\}$. Write all elements of ρ, its domain and range. Plot ρ on a coordinate diagram.

2. Let R be the set of real numbers and ρ the relation defined by

$$x \rho y \text{ iff both } x \in [n, n+1] \text{ and } y \in [n, n+1]$$

for some integer n.

Plot the relation ρ.

SOLUTION:

We shall begin with a definition:

DEFINITION

A relation ρ is a subset $\rho \subset A \times B$. Often, $(a, b) \in \rho$ is written $a \rho b$. ■

1. A relation ρ consists of ordered pairs $(a, b) \in A \times B$, such that $a < b$. Therefore,

$$\rho = \{(1, 3), (1, 4), (1, 5), (3, 4), (3, 5), (4, 5)\}. \tag{1}$$

Note that the domain of ρ is the set of the first coordinates of the pairs in ρ. From (1) we have the

$$\text{domain of } \rho = \{1, 3, 4\}. \tag{2}$$

Similarly, the range of a relation is the set of the second coordinates of the pairs in this relation.

Hence,

$$\text{range of } \rho = \{3, 4, 5\}. \tag{3}$$

Figure 1 depicts the sets A and B and the relation <.

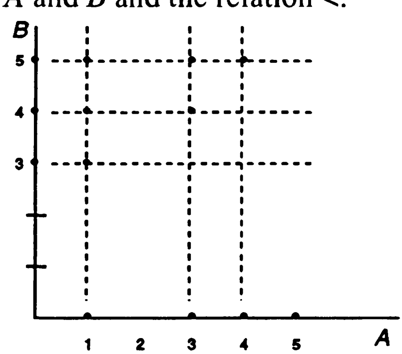


FIGURE 1.

2. Suppose $n = 0$, then

$$x \rho y \text{ if } x \in [0, 1] \quad \text{and} \quad y \in [0, 1],$$

which is a square shown in Figure 2.

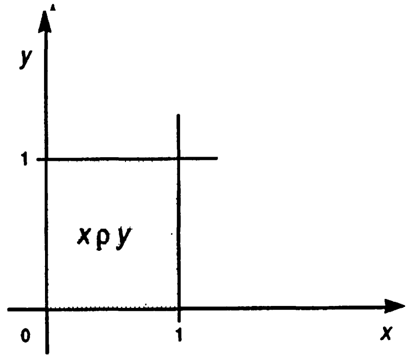


FIGURE 2.

For each integer n we obtain a square as shown in Figure 3.

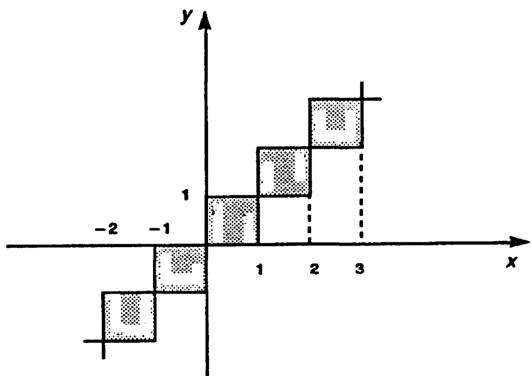


FIGURE 3.

● PROBLEM 3–14

1. Let Z denote the set of all integers. Find the diagonal of Z.

2. Let N denote the set of natural numbers and ρ, a relation defined by

$$n \rho m \text{ iff } n = 3m + 1. \tag{1}$$

Find the inverse of ρ, denoted by ρ^{-1}.

SOLUTION:

1. For any set A, its diagonal Δ is the relation of equality defined by

$$\Delta = \{(a, a) : a \in A\} \subset A \times A. \tag{2}$$

Thus, if Z is the set of all integers, its diagonal is the set

$$\Delta = \{(n, n) : n \in Z\} =$$

$$= \{\ldots (-2, -2), (-1, -1), (0, 0), (1, 1), (2, 2), \ldots\}. \tag{3}$$

2. Let ρ denote a relation defined as $A \times B$. The inverse of ρ, denoted by ρ^{-1}, is the subset of $B \times A$, defined by

$$\rho^{-1} = \{(b, a) : (a, b) \in \rho\}. \tag{4}$$

Relation ρ, defined by (1), consists of all the pairs

$$\rho = \{(1, 0), (4, 1), (7, 2), (10, 3), (-2, -1), (-5, -2) \ldots\} \tag{5}$$

such that

$$(n, m) \in \rho \text{ if, and only if } n = 3m + 1. \tag{6}$$

The inverse ρ^{-1} consists of all the pairs $(m, n) \in \rho^{-1}$, such that

$$(m,n) \ \varepsilon \ \rho^{-1} \text{ iff } m = \frac{n-1}{3}. \tag{7}$$

That leads to

$$\rho^{-1} = \{(0, 1), (1, 4), (2, 7), (3, 10), (-1, -2), (-2, -5) \ldots\} \tag{8}$$

● PROBLEM 3–15

Consider three sets A, B, and C.

$$A = \{1, 2, 3, 4\}$$

$$B = \{5, 6, 7, 8\}$$

$$C = \{x, y, z\}. \tag{1}$$

Let ρ be a relation on $A \times B$

$$\rho = \{(1, 6), (1, 7), (3, 6), (3, 7), (3, 8), (4, 5)\} \tag{2}$$

and λ be a relation on $B \times C$

$$\lambda = \{(5, x), (6, y), (7, y), (7, z), (8, x)\}. \tag{3}$$

Find the composition of ρ and λ denoted by $\lambda \circ \rho$ (sometimes the symbol $\rho \circ \lambda$ is used instead).

SOLUTION:

We shall start with a definition of a composition of relations. Let A, B, C represent the sets, and $\rho \subset A \times B$ and $\lambda \subset B \times C$ represent two relations. The composition of ρ and λ, denoted by $\lambda \circ \rho$, is the relation on $A \times C$, such that

$$\lambda \circ \rho = \{(a, c) : a \in A,\ c \in C,\ \exists b \in B \text{ such that}$$

$$(a, b) \in \rho \wedge (b, c) \in \lambda\}. \tag{4}$$

Therefore, from (2), (3), and definition (4), we find

$$(1, y) \in \lambda \circ \rho \text{ because } (1, 6) \in \rho \text{ and } (6, y) \in \lambda$$

$$(4, x) \in \lambda \circ \rho \text{ because } (4, 5) \in \rho \text{ and } (5, x) \in \lambda$$

$$(1, z) \in \lambda \circ \rho \text{ because } (1, 7) \in \rho \text{ and } (7, z) \in \lambda$$

$$(3, y) \in \lambda \circ \rho \text{ because } (3, 6) \in \rho \text{ and } (6, y) \in \lambda$$

$$(3, z) \in \lambda \circ \rho \text{ because } (3, 7) \in \rho \text{ and } (7, z) \in \lambda$$

$$(3, x) \in \lambda \circ \rho \text{ because } (3, 8) \in \rho \text{ and } (8, x) \in \lambda$$

$$(4, x) \in \lambda \circ \rho \text{ because } (4, 5) \in \rho \text{ and } (5, x) \in \lambda$$

The composition $\lambda \circ \rho$ is the set consisting of

$$\lambda \circ \rho = \{(1, y), (4, x), (1, z), (3, y), (3, z), (3, x), (4, x)\}. \tag{5}$$

The relation $\lambda \circ \rho$ is illustrated in Figure 1.

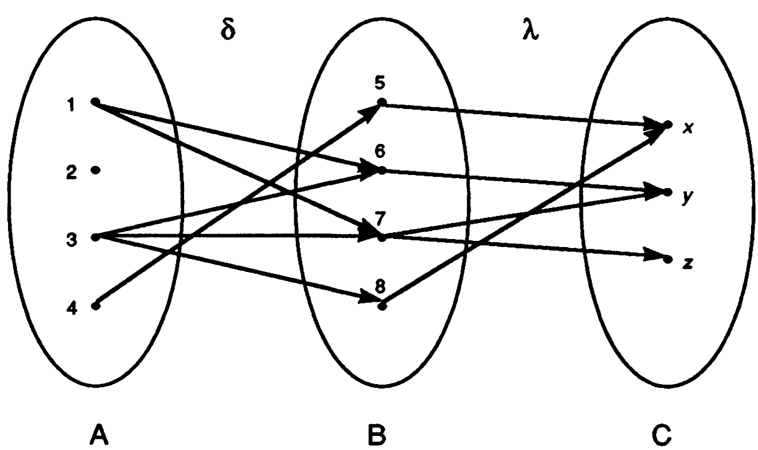


FIGURE 1.

For example $(3, x) \in \lambda \circ \rho$ because there is a "route" from 3 to x via 8.

Let A, B, C, and D represent the sets and $\rho \subset A \times B$, $\lambda \subset B \times C$, and $\delta \subset C \times D$ represent the relations. Prove that:

1. $(\delta \circ \lambda) \circ \rho = \delta \circ (\lambda \circ \rho)$ (1)

2. $(\lambda \circ \rho)^{-1} = \rho^{-1} \circ \lambda^{-1}$ (2)

SOLUTION:

1.

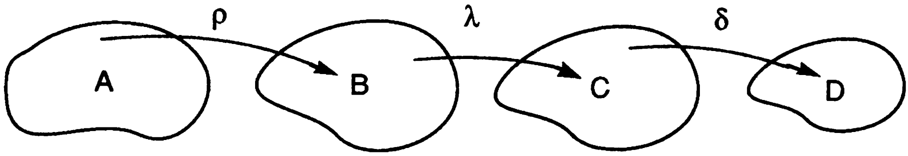


FIGURE 1.

$$(\delta \circ \lambda) \circ \rho = \{(a, d) : a \in A \wedge d \in D, \exists b \in B \text{ such that}$$

$$(a, b) \in \rho \wedge (b, d) \in \delta \circ \lambda\} =$$

$$= \{(a, d) : \exists b \in B, \exists c \in C \text{ such that}$$

$$(a, b) \in \rho, (b, c) \in \lambda, (c, d) \in \delta\} =$$

$$= \{(a, d) : \exists c \in C \text{ such that } (a, c) \in \lambda \circ \rho, (c, d) \in \delta\} =$$

$$= \delta \circ (\lambda \circ \rho). \tag{3}$$

Hence, we can write

$$(\delta \circ \lambda) \circ \rho = \delta \circ (\lambda \circ \rho) = \delta \circ \lambda \circ \delta. \tag{4}$$

2. $$(\lambda \circ \rho)^{-1} = \{(c, a) : (a, c) \in \lambda \circ \rho\} =$$

$$= \{(c, a) : \exists b \in B, \text{ such that } (a, b) \in \rho, (b, c) \in \lambda\} =$$

$$= \{(c, a) : \exists b \in B, \text{ such that } (b, a) \in \rho^{-1}, (c, b) \in \lambda^{-1}\} =$$

$$= \{(c, a) : \exists b \in B, \text{ such that } (c, b) \in \lambda^{-1}, (b, a) \in \rho^{-1}\} =$$

$$= \rho^{-1} \circ \lambda^{-1}. \tag{5}$$

Similarly,

$$(\delta \circ \lambda \circ \rho)^{-1} = [\delta \circ (\lambda \circ \rho)]^{-1} =$$

$$= (\lambda \circ \rho)^{-1} \circ \delta^{-1} = \rho^{-1} \circ \lambda^{-1} \circ \delta^{-1}. \tag{6}$$

• PROBLEM 3–17

Let A represent the set of all people and R represent a relation defined as $A \times A$, such that for $a, b \in A$ and $a\ R\ b$ iff a knows b; that is, a is in relation to b if and only if a knows b. Is this relation an equivalence relation?

SOLUTION:

An equivalence relation is a very useful and frequently applied concept in mathematics. It enables us to divide the set into subsets (classes) according to some preassigned characteristics.

DEFINITION OF EQUIVALENCE RELATION

A relation R in A is called an equivalence relation if:

1. It is reflexive: $\forall a \in A,\ a\ R\ a$

2. It is symmetric: $(a\ R\ b) \Rightarrow (b\ R\ a)$

3. It is transitive: $(a\ R\ b) \wedge (b\ R\ c) \Rightarrow (a\ R\ c)$

If $a\ R\ b$, we say that a and b are equivalent.

■

The relation "knows" is reflexive, since obviously a knows a.

It is symmetric, because if a knows b, then b knows a.

It is not transitive, because if a knows b and b knows c, it does not guarantee that a knows c.

Therefore, the relation "knows" is not an equivalence relation.

Prove the following:

Let R denote a relation in A, that is, $R \subset A \times A$.

Then:

1. R is reflexive iff $\Delta \subset R$ (where Δ is the diagonal of A).

2. R is symmetric iff $R = R^{-1}$.

3. R is transitive iff $R \circ R \subset R$.

4. (R is symmetric) $\Rightarrow$ ($R \circ R^{-1} = R^{-1} \circ R$)

5. (R is reflexive) $\Rightarrow$ $\begin{pmatrix} \text{1. } R \subset R \circ R \\ \text{2. } R \circ R \text{ is reflexive} \end{pmatrix}$

6. (R is transitive) $\Rightarrow$ ($R \circ R$ is transitive)

SOLUTION:

1. The diagonal of A is

$$\Delta = \{(a, a) : a \in A\}. \tag{1}$$

Thus R is reflexive iff,

$$\forall a \in A, \quad (a, a) \in R$$

iff $\Delta \subset R$.

2. (R – symmetric) $\Leftrightarrow$ ($R = R^{-1}$)
$\Rightarrow$ If $(a, b) \in R$, then $(b, a) \in R$, therefore $R = R^{-1}$.
$\Leftarrow$ Obvious.

3. Let $(a, c) \in R \circ R$. Then $\exists b \in A$, such that $(a, b) \in R$ and $(b, c) \in R$. Since R is transitive, and (a, b), $(b, c) \in R$ implies that $(a, c) \in R$, hence $R \circ R \subset R$.

Conversely, if $R \circ R \subset R$, then $(a, b) \in R$ and $(b, c) \in R$ imply that $(a, c) \in R \circ R \subset R$. That is, R is transitive.

4. $R \circ R^{-1} = \{(a, c) : \exists b \in A, \text{ such that } (a, b) \in R^{-1} \wedge (b, c) \in R\} =$

$$= \{(a, c) : \exists b \in A, (a, b) \in R \wedge (b, c) \in R^{-1}\} =$$

$$= R^{-1} \circ R. \tag{2}$$

5. Let $(a, b) \in R$. Then

$$R \circ R = \{(a, c) : \exists b \in A, (a, b) \in R \wedge (b, c) \in R\}. \tag{3}$$

But, $(a, b) \in R$ and since R is reflexive, $(b, b) \in R$. Hence $(a, b) \in R \circ R$, that is, $R \subset R \circ R$.

Also,

$$\Delta \subset R \subset R \circ R \text{ implies that } R \circ R \text{ is reflexive.}$$

6. Let $(a, b) \in R \circ R$ and $(b, c) \in R \circ R$. Then from (3), $R \circ R \subset R$. Thus, $(a, b), (b, c) \in R$. So $(a, c) \in R \circ R$, and $R \circ R$ is transitive.

● PROBLEM 3–19

The set A consists of

$$A = \{1, 2, 3, 4, \alpha, \beta\}. \tag{1}$$

Find the smallest equivalence relation R, such that

$$(1, 2) \in R, \quad (2, \alpha) \in R, \quad (4, 2) \in R. \tag{2}$$

SOLUTION:

Relation R has to be reflexive, therefore,

$$(1, 1) \in R,\ (2, 2) \in R,\ \ (3, 3) \in R,\ (4, 4) \in R,$$

$$(\alpha, \alpha) \in R, \quad (\beta, \beta) \in R. \tag{3}$$

It has to be symmetric, hence

$$(2, 1) \in R, \quad (\alpha, 2) \in R, \quad (2, 4) \in R. \tag{4}$$

Finally, R has to be transitive, thus,

$$(1, 2) \in R \wedge (2, \alpha) \in R \Rightarrow (1, \alpha) \in R$$

$$(1, 2) \in R \wedge (4, 2) \in R \Rightarrow (1, 4) \in R$$

$$(2, \alpha) \in R \wedge (4, 2) \in R \Rightarrow (4, \alpha) \in R. \tag{5}$$

The smallest equivalence relation is

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (\alpha, \alpha), (\beta, \beta), (1, 2), (2, \alpha), (4, 2),$$

$$(2, 1), (\alpha, 2), (2, 4), (1, \alpha), (\alpha, 1), (1, 4), (4, 1), (4, \alpha), (\alpha, 4)\}. \tag{6}$$

The largest equivalence relation consists of all 36 elements of $A \times A$.

● PROBLEM 3-20

Let Z^+ denote the set of positive integers,

$$Z^+ = \{1, 2, \ldots\} \tag{1}$$

and let ρ represent the relation in $Z^+ \times Z^+$ defined by

$$(a, b)\ \rho\ (c, d) \text{ iff } a + d = b + c. \tag{2}$$

Show that ρ is an equivalence relation.

SOLUTION:

Element (a, b) is in relation ρ with itself. Indeed

$$(a, b)\ \rho\ (a, b) \text{ because } a + b = b + a. \tag{3}$$

Thus, ρ is reflexive.

Suppose:

$$(a, b)\ \rho\ (c, d), \text{ then } a + d = b + c \text{ or}$$

$$c + b = d + a. \tag{4}$$

Hence, $(c, d)\ \rho\ (a, b)$ and ρ are symmetric.

Suppose:

$$(a, b)\ \rho\ (c, d) \quad \text{and} \quad (c, d)\ \rho\ (e, f) \tag{5}$$

then

$$a + d = b + c \quad c + f = d + e. \tag{6}$$

Hence,

$$a + d + c + f = b + c + d + e \tag{7}$$

and

$$a + f = b + e. \tag{8}$$

From (8) we conclude that

$$(a, b)\ \rho\ (e, f). \tag{9}$$

Relation ρ is transitive. Thus, ρ is an equivalence relation.

● PROBLEM 3-21

Let $Z^+ \times Z^+$ denote the set of ordered pairs of positive integers. Let ρ denote the relation in $Z^+ \times Z^+$ defined by

$$(a, b)\ \rho\ (c, d) \text{ iff } ad = bc. \tag{1}$$

Show that ρ is an equivalence relation. Find the equivalence class of (1,1).

SOLUTION:

Relation ρ is reflexive, because for every $(a, b) \in Z^+ \times Z^+$

$$(a, b)\ \rho\ (a, b), \text{ that is, } ab = ba. \tag{2}$$

Relation ρ is symmetric.

Suppose

$$(a, b)\ \rho\ (c, d), \text{ that is, } ad = bc, \tag{3}$$

and $cb = da$. Therefore,

$$(c, d)\ \rho\ (a, b) \tag{4}$$

and ρ is symmetric.

Suppose

$$(a, b)\ \rho\ (c, d) \quad \text{and} \quad (c, d)\ \rho\ (e, f). \tag{5}$$

That is,

$$ad = cb \ \text{ and } \ cf = de. \tag{6}$$

Hence,

$$adcf = cbde \tag{7}$$

and

$$af = be. \tag{8}$$

Therefore, $(a, b)\ \rho\ (e, f)$ and ρ is transitive.

Relation ρ is reflexive, symmetric and transitive, thus it is an equivalence relation.

Suppose (a, b) is in relation ρ with $(1, 1)$

$$(a, b)\ \rho\ (1, 1) \quad \text{iff} \quad a = b. \tag{9}$$

The equivalence class of $(1, 1)$ consists of all pairs (a, a), $a \in Z^+$.

• PROBLEM 3-22

Consider a set X and its partition into disjoint classes

$$X = \bigcup_{i \in I} A_i \,. \tag{1}$$

This partition defines a relation $R \subset X \times X$, in such a way that $(x, y) \in R$ when $\exists i : x, y \in A_i$, that is, when the elements x, y belong to the same Class A_i. Show that R is an equivalence relation.

SOLUTION:

R is reflexive, because $(x, x) \in R$ means that x belongs to a certain class A_i, which is true, since

$$X = \bigcup_i A_i$$

That R is symmetric is obvious, because if $x, y \in A_i$, then $y, x \in A_i$. Hence

$$(x, y) \in R \Rightarrow (y, x) \in R. \tag{2}$$

Suppose $(x, y) \in R \wedge (y, z) \in R$. That means

$$\exists i\ \exists j\ (x, y \in A_i) \wedge (y, z \in A_j). \tag{3}$$

Thus

$$y \in A_i \cap A_j \neq 0 \tag{4}$$

which is a contradiction, because the classes are disjoint.

We proved that relation R is an equivalence.

In the next problem we shall prove the converse.

• PROBLEM 3-23

Prove the following:

THEOREM

Any equivalence relation in X defines a partition of X into classes of the form

$$A_x = \{y \in X : (x, y) \in R\} \tag{1}$$

■

SOLUTION:

Suppose R is an equivalence relation in X. Therefore R is reflexive

$$\forall x \in X \ (x, x) \in R \ \text{ and } \ x \in A_x. \tag{2}$$

Hence

$$X = \bigcup_x A_x \,. \tag{3}$$

The classes cover the whole set X. We shall show that the classes are disjoint.

Suppose

$$A_x \cap A_y \neq \phi. \tag{4}$$

Then z exists, such that $z \in A_x \wedge z \in A_y$

$$[(z, x) \in R \wedge (z, y) \in R] \Rightarrow (x, y) \in R. \tag{5}$$

Therefore,

$$A_x = A_y. \tag{6}$$

That completes the proof.

The set of classes

$$\{A_x : x \in X\} \tag{7}$$

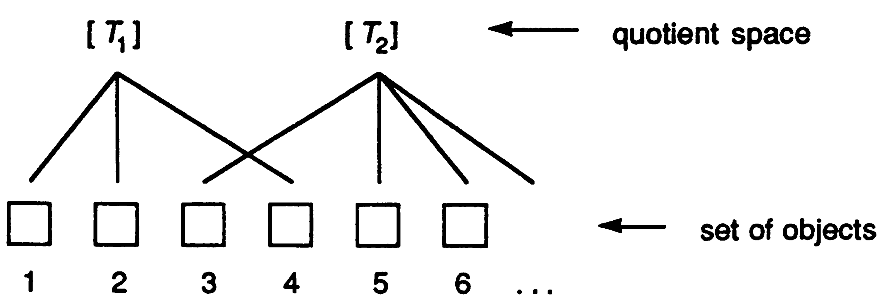
is called a quotient space and is denoted by $^X/_R$. Mapping

$$\zeta : X \to \frac{X}{R} \tag{8}$$

where $\zeta(x) = A_x$, is called a canonical mapping or projection. The class A_x is denoted by $[x]$.

The principle of partition of a set into classes forms a basis of abstract thinking – one of the most interesting features of the human mind.

For example, abstract thinking leads us to the concept of temperature. Let X represent the set of all objects.



By using a thermometer, we establish an equivalence relation.

Two objects x and y are in relation T with each other, if they have the same temperature,

$$x \, T \, y. \tag{9}$$

It is easy to see that this is an equivalence relation. Neglecting all other properties like color, weight, age, etc., we classify objects into classes of equal temperature.

● PROBLEM 3–24

Let $\zeta(x, y)$ represent a given sentence function of two variables. Thus,

$$\forall y : \zeta(x, y) \text{ and } \exists y : \zeta(x, y) \tag{1}$$

are sentence functions of one variable. Show that operation $\forall$ is commutative with respect to $\forall$, and $\exists$ is commutative with respect to $\exists$. That is, show that

$$\forall x \, \forall y \, \zeta(x, y) \equiv \forall y \, \forall x \, \zeta(x, y) \tag{2}$$

$$\exists x \, \exists y \, \zeta(x, y) \equiv \exists y \, \exists x \, \zeta(x, y) \tag{3}$$

Hence, the sequence of quantifiers is immaterial in (2) and (3).

On the other hand, the sequence of quantifiers $\forall$ and $\exists$ is significant. Prove the formula:

$$\exists x \, \forall y \, \zeta(x, y) \Rightarrow \forall y \, \exists x \, \zeta(x, y). \tag{4}$$

SOLUTION:

The sentence $\zeta(x, y)$ is true for all x and y, therefore, it is true for all y and x.

Similarly, if x and y exist, such that $\zeta(x, y)$ is true, then y and x exist, such that $\zeta(x, y)$ is true.

The left-hand side of (4) indicates that x_0 exists such that, for every y, $\zeta(x_0, y)$ is true. Therefore, for every y, an x (namely $x = x_0$) exists, such that $\zeta(x, y)$ is true. Hence, the left-hand side implies the right-hand side.

1. Show that the implication (4) of Problem 3–24 is not valid in the opposite direction.

2. Prove the formulas:

$$\exists x\,[\zeta(x) \wedge \varphi(x)] \Rightarrow \exists x\,\exists y\,[\zeta(x) \wedge \varphi(y)] \equiv$$

$$\equiv \exists x : \zeta(x) \wedge \exists y : \varphi(y) \equiv \exists x : \zeta(x) \wedge \exists x : \varphi(x) \qquad (1)$$

and

$$\forall x : \zeta(x) \vee \forall x : \varphi(x) \equiv \forall x, y\,[\zeta(x) \vee \varphi(y)] \Rightarrow$$

$$\Rightarrow \forall x\,[\zeta(x) \vee \varphi(x)]. \qquad (2)$$

SOLUTION:

1. Take, for instance, the set of real numbers. It is true that

$$\forall y\,\exists x : y < x. \qquad (3)$$

On the other hand, the statement

$$\exists x\,\forall y : y < x \qquad (4)$$

is not true. Hence, the implication in the opposite direction is not true.

2. Obviously the implication

$$\forall x : \zeta(x) \Rightarrow \exists x : \zeta(x) \qquad (5)$$

is true. Setting $X = Y$, we can replace (5) by

$$\forall x\,\forall y : \zeta(x, y) \Rightarrow \forall x : \zeta(x, x) \Rightarrow \exists x : \zeta(x, x) \Rightarrow \exists x\,\exists y : \zeta(x, y). \qquad (6)$$

Similarly, from

$$[\exists x : \zeta(x) \wedge \varphi(x)] \Rightarrow \exists x : \zeta(x) \wedge \exists x : \varphi(x) \qquad (7)$$

we obtain

$$\exists x\,[\zeta(x) \wedge \varphi(x)] \Rightarrow \exists x\,\exists y\,[\zeta(x) \wedge \varphi(y)]\,. \qquad (8)$$

The right-hand side is equivalent to

$$\exists x\,\exists y\,[\zeta(x) \wedge \varphi(y)] \equiv \exists x : \zeta(x) \wedge \exists y : \varphi(y) \equiv$$

$$\equiv \exists x : \zeta(x) \wedge \exists x : \varphi(x). \qquad (9)$$

Similarly, from

$$[\forall x : \zeta(x) \vee \forall x : \varphi(x)] \Rightarrow \forall x\, [\zeta(x) \vee \varphi(x)] \tag{10}$$

we obtain (2).

● PROBLEM 3–26

In Chapter 2, we discussed the axiomatic formulation of the algebra of sets. Then, four axioms:

1. Uniqueness Axiom

2. Union Axiom

3. Difference Axiom

4. Existence Axiom

were sufficient. What additional axioms are necessary for the purposes of Chapter 3?

SOLUTION:

We should add the following axioms:

5. For every sentence function $\zeta(x)$ and for every set A, a set exists consisting of those, and only those elements of A for which $\zeta(x)$ is true.
This set is denoted by

$$\{x : \zeta(x), x \in A\}.$$

Without realizing it, we used this axiom frequently.
For example, Axiom 5 guarantees the existence of the set $\{a, b\}$

$$\{a, b\} = \{x : x = a \vee x = b, x \in A\}.$$

6. For every set A, a set, whose elements are all the subsets of A, exists. This set is denoted by 2^A.

7. **AXIOM OF CHOICE.**
For every family R of non-empty disjoint sets, a set exists which has one, and only one, element in common with each of the sets of the family R.

The last axiom is an existential axiom. In general, a set obtained by application of this axiom is not uniquely determined. In 1938, K. Gödel showed that if the axiomatic set theory without axiom of choice is consistent, then it is also consistent *with* the axiom of choice.

For some time, some mathematicians suspected that axiom of choice can be derived from the other axioms. In 1963, P.J. Cohen proved that this is not the case.

The axiom of choice is an independent axiom.

CHAPTER 4

MAPPINGS

Which of the diagrams depicts a function from $X = \{a, b, c\}$ to $Y = \{x, y, z\}$?

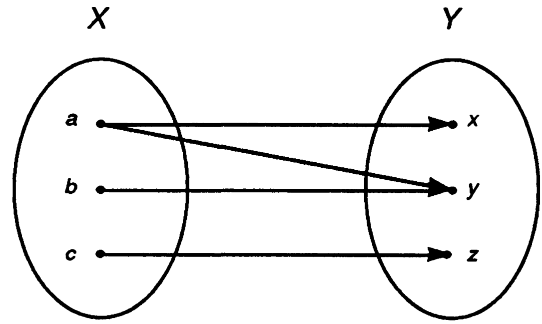


FIGURE 1

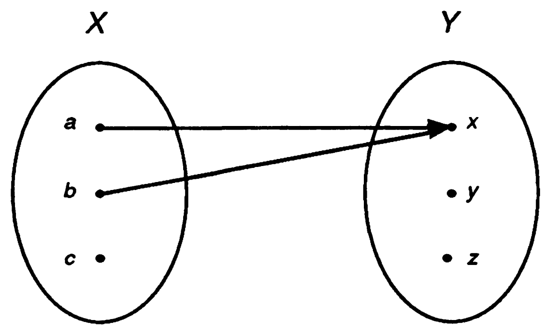


FIGURE 2

SOLUTION:

DEFINITION

f is a function from X to Y iff $f \subseteq X \times Y$, for all $x \in X$, there is a $y \in Y$, such that $(x, y) \in f$; $(x, y_1) \in f$ and $(x, y_2) \in f$ imply $y_1 = y_2$.

Function f is denoted

$$f : X \rightarrow Y. \tag{1}$$

The domain of f is X, the co-domain is Y.

The element $f(x) = y \in Y$ is called the image of x under f. Figure 1 illustrates the relation consisting of the pairs

$$(a, x), (a, y), (b, y), (c, z).$$

This relation assigns two elements, $x \in Y$ and $y \in Y$, to element $a \in X$. Hence, it is not a function.

Figure 2 also does not define a function. This relation is not defined for all elements of X. Element c is left "alone." It is easy to upgrade the relation shown in Figure 2 to obtain a function as shown in Figure 3.

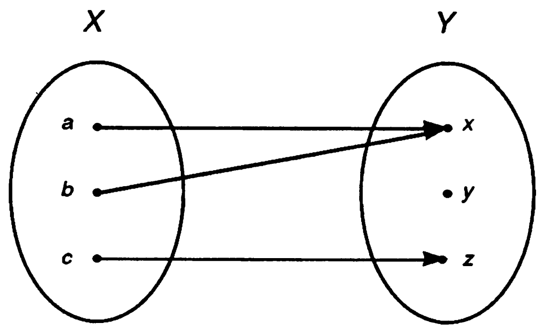


FIGURE 3

Note that a function $f : X \to Y$ has to be defined for all elements of X, but it does not have to "cover" all elements of Y, as shown in Figure 3. In the extreme case, a function f can assign all elements of X to one element of Y.

Sometimes a word mapping or map is used instead of a function.

● PROBLEM 4–2

Prove that not every injection of a set into itself is a bijection.

SOLUTION:

We shall start with some definitions.

DEFINITION (SURJECTION (ONTO) FUNCTION)

A function f from X to Y, $f : X \to Y$ is an onto function (also called surjection) if, and only if, for all $y \in Y$, an $x \in X$ exists such that $y = f(x)$.

For example, function $f : R \to R$, where R is the set of real numbers, is a surjection when $f(x) = 3x + 1$ and is not a surjection when $f(x) = x^2$.

DEFINITION (INJECTION – ONE-TO-ONE FUNCTION)

A function $f : X \to Y$ is a one-to-one function (also called injection) iff $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

DEFINITION (BIJECTION)

A function $f : X \to Y$ is a bijection iff f is both a surjection and injection.

For example, the function shown in Figure 1

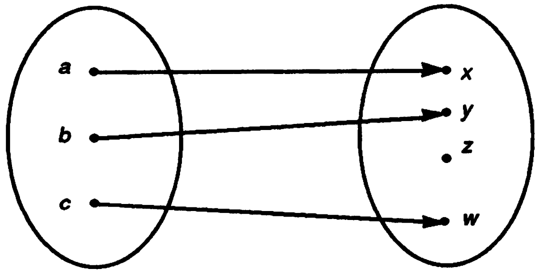


FIGURE 1

is one-to-one, but it is not onto, hence it is not a bijection.

We shall find function $f : X \to X$ which is an injection, but not a surjection. Let Z^+ denote the set of positive integers, then

$$f : Z^+ \to Z^+$$

such that

$$f(n) = n + 1$$

is one-to-one but not onto. Indeed no positive integer n exists such that

$$f(n) = 1.$$

Hence, $f(n) = n + 1$ is not a bijection.

● PROBLEM 4–3

1. Find the graph of the function shown in Figure 1.

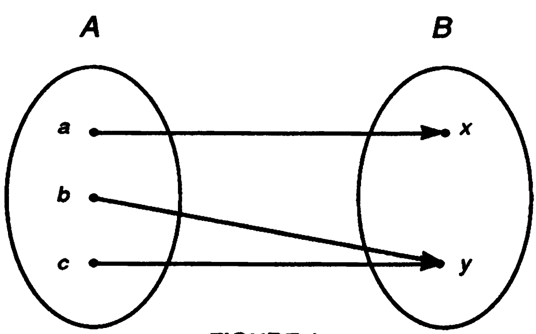


FIGURE 1

2. Let $f : R \to R$ and $g : R \to R$, where $f(x) = (x + 1)^2$, and $g(x) = x^2 + 2x + 1$ and R is the set of real numbers..Are these functions equal?

3. Define the restriction and extension of a function.

SOLUTION:

1. The set G

$$G = \{(x, y) : f(x) = y\} \subset X \times Y \tag{1}$$

is called the graph of f. From Figure 1, we find the graph of the function to be

$$\{(a, x), (b, y), (c, y)\}. \tag{2}$$

2. **DEFINITION**

Two functions $f : A \to B$ and $g : A \to B$ are equal, $f = g$ iff

$$f(a) = g(a) \text{ for every } a \in A.$$

Obviously, for all real numbers

$$(x + 1)^2 = x^2 + 2x + 1. \tag{3}$$

Hence $f(x) = g(x)$.

3. **DEFINITION**

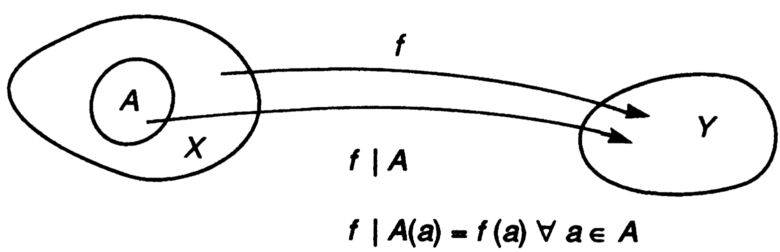
Let $f : X \to Y$ and $A \subset X$; the function f considered only on A is called the restriction of f to A, written $f|A$, if

$$f|A = f \cap (A \times Y). \tag{4}$$

Similarly, if $A \subset X$ and $g : A \to Y$ is a given function, then

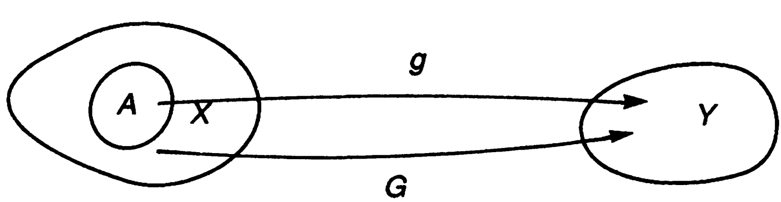
$$G : X \to Y$$

such that $G|A = g$ is called an extension of g over X.



$f|A$ is the restriction of f to A

FIGURE 2



G is an extension of g.

FIGURE 3

● PROBLEM 4–4

Let f^{-1} be the inverse of the function f. In general, f^{-1} is only a relation. Find the conditions for f^{-1} to be a function.

SOLUTION:

We defined a function $f: X \to Y$ as a subset of $X \times Y, f \subset X \times Y$, such that

$$[(x, y_1) \in f \wedge (x, y_2) \in f] \Rightarrow [y_1 = y_2]. \qquad (1)$$

Since f is a relation, we can always find f^{-1}.

Function f must be a surjection, that is, $f(X) = Y$, then f^{-1} is defined for all $y \in Y$. In order for f^{-1} to be a function, function f must be an injection. Indeed, if

$$[(x_1, y) \in f \wedge (x_2, y) \in f] \Rightarrow [x_1 = x_2]$$

then we have

$$[(y, x_1) \in f' \wedge (y, x_2) \in f'] \Rightarrow [x_1 = x_2].$$

We conclude that only a function, which is a bijection, has the inverse function. It is easy to show that

$$(f^{-1})^{-1} = f.$$

● PROBLEM 4–5

Let the functions

$$f: R \to R \text{ and } g: R \to R$$

be defined by

$$f(x) = 3x - 1 \quad g(x) = x^3 + 1.$$

Find their compositions

$$f \circ g \quad \text{and} \quad g \circ f.$$

SOLUTION:

DEFINITION OF A COMPOSITION

Let $f: A \to B$ and $g: B \to C$. The composition of f and g is the function $g \circ f$ defined by:

$$(g \circ f)(a) = g(f(a)). \qquad (1)$$

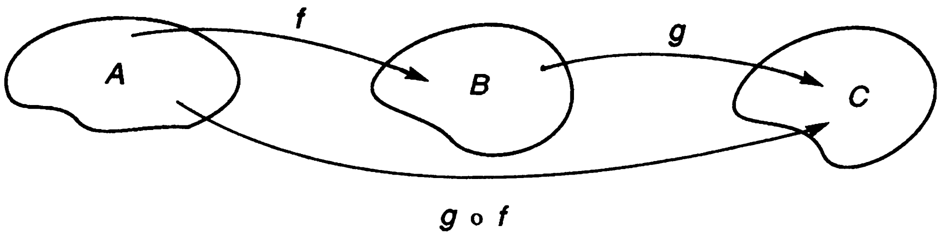


FIGURE 1

Let us find the composition $g \circ f$

$$(g \circ f)(x) = g(f(x)) = g(3x - 1) = (3x - 1)^3 + 1. \tag{2}$$

Similarly,

$$(f \circ g)(x) = f(g(x)) = f(x^3 + 1) = 3(x^3 + 1) - 1. \tag{3}$$

From (2) and (3) we conclude that

$$f \circ g \neq g \circ f. \tag{4}$$

● PROBLEM 4-6

Prove that the composition of functions is associative. That is, prove that if

$$f : A \to B, \quad g : B \to C, \quad h : C \to D$$

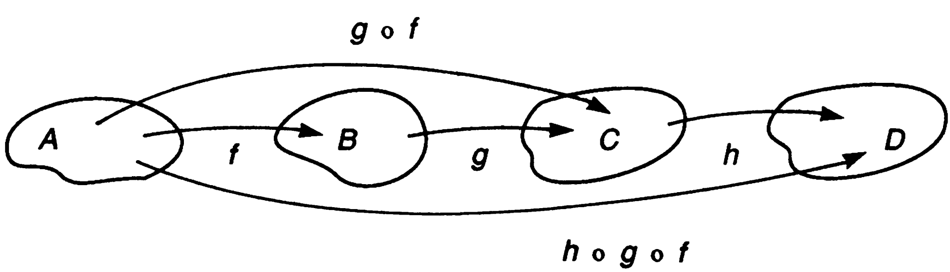


FIGURE 1

then

$$(h \circ g) \circ f \ = \ h \circ (g \circ f). \tag{1}$$

SOLUTION:

In Chapter 3 we proved that relations are associative and since functions are relations, this result holds for functions as well. Here is another way of showing it:

$$[(h \circ g) \circ f\,](a) = (h \circ g)\,[f(a)] =$$

$$= h[g(f(a))] \quad \text{for every } a \in A. \tag{2}$$

Similarly,

$$[h \circ (g \circ f)\,](a) = h\,[(g \circ f)\,(a)] =$$

$$= h[g(f(a))] \quad \text{for every } a \in A. \tag{3}$$

Thus

$$(h \circ g) \circ f = h \circ (g \circ f) = h \circ g \circ f. \tag{4}$$

● PROBLEM 4-7

1. Let $f : A \to B$, $g : B \to C$. Prove that

   a) If f and g are onto, then

   $$g \circ f : A \to C \text{ is onto;}$$

   b) If f and g are one-to-one, then

   $$g \circ f \text{ is one-to-one.}$$

2. Determine which of the following functions are onto, one-to-one or bijection.

$$f : A \to A, \quad A = [-1, 1]$$

$$f(x) = x^2, \quad f(x) = \sin x$$

$$f(x) = x^3, \quad f(x) = \sin \pi x \tag{1}$$

SOLUTION:

1. We shall show that $g \circ f$ is onto when both g and f are onto.
   Let $c \in C$, since g is onto

$$\exists b \in B, \text{ such that } g(b) = c.$$

Also, f is onto, therefore,

$$\exists a \in A, \text{ such that } f(a) = b.$$

We conclude

$$\forall c \in C \quad \exists a \in A, \text{ such that } (g \circ f)(a) = c \tag{2}$$

hence $g \circ f$ is onto.

Now we show that if f and g are one-to-one, then $g \circ f$ is one-to-one. Suppose it is not the case

$$(g \circ f)(a) = (g \circ f)(a') \tag{3}$$

or

$$g(f(a)) = g(f(a')). \tag{4}$$

But g is one-to-one. Hence

$$f(a) = f(a'). \tag{5}$$

Also f is one-to-one, hence

$$a = a'.$$

We conclude that $g \circ f$ is one-to-one.

2. Function $f(x) = x^2$ (Figure 1) is

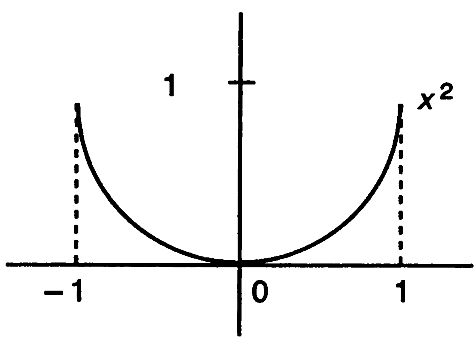


FIGURE 1

not onto because $f([-1, 1]) = [0,1]$. It is not one-to-one because, for example,

$$(-1)^2 = (1)^2.$$

Function $f(x) = \sin x$ (Figure 2) is one-to-one, but is not onto.

Function $f(x) = x^3$ (Figure 3) is one-to-one and onto, hence it is a bijection.

Function $f(x) = \sin \pi x$ (Figure 4) is onto but not one-to-one.

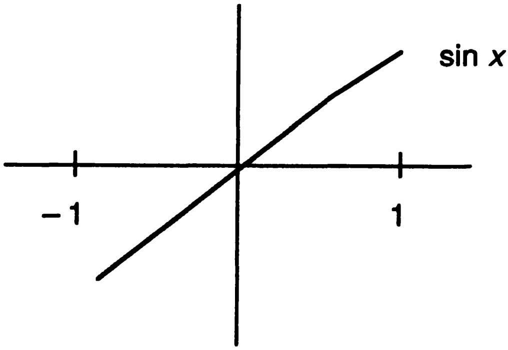


FIGURE 2

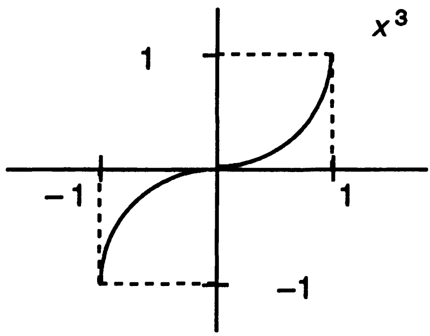


FIGURE 3

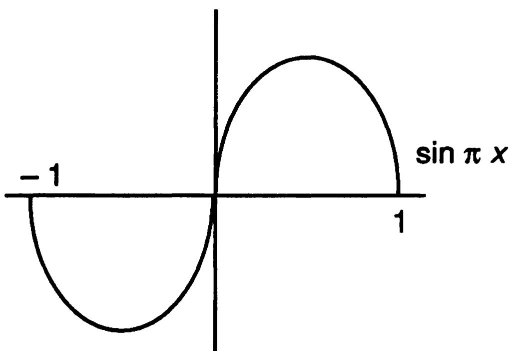


FIGURE 4

● PROBLEM 4–8

Prove that:
If $f: A \rightarrow B$ and $g: B \rightarrow C$ are one-to-one and onto, then

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}. \tag{1}$$

SOLUTION:

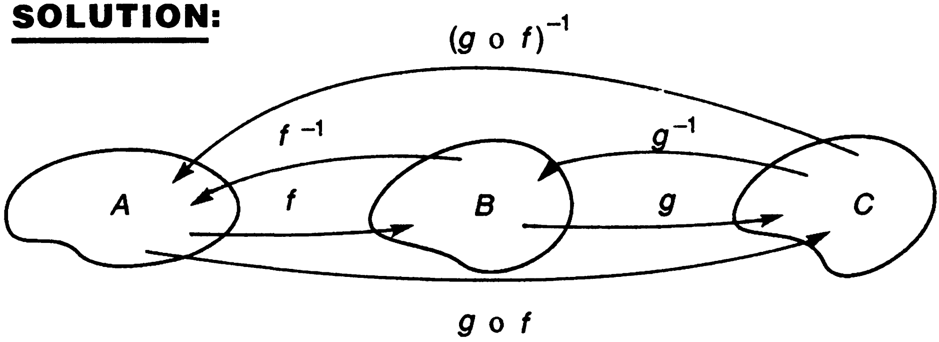


FIGURE 1

Let 1_A be the identity function on A

$$1_A(a) = a, \forall a \in A. \tag{2}$$

By applying the associative law of composition of functions, we find

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ [g^{-1} \circ (g \circ f)] =$$

$$= f^{-1} \circ [(g^{-1} \circ g) \circ f] = f^{-1} \circ (1 \circ f) =$$

$$= f^{-1} \circ f = 1_A \tag{3}$$

since $g^{-1} \circ g = 1_A$ and $1_A \circ f = f \circ 1_A = f$.

Similarly,

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ [f \circ (f^{-1} \circ g^{-1})] =$$

$$= g \circ [(f \circ f^{-1}) \circ g^{-1}] =$$

$$= g \circ (1 \circ g^{-1}) = g \circ g^{-1} = 1_A. \tag{4}$$

Therefore,

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}. \tag{5}$$

● PROBLEM 4-9

1. Let $X = [-1, 1]$ and $Y = R$, and

$$f : X \to R \text{ be } f(x) = x^2. \tag{1}$$

Find the image of X in R under f.

2. Find the inverse image of $[1, 4]$ in X under f, $f(x) = x^2$.

3. Prove that:

If $f : X \to Y$ and $A \subset X$, $B \subset X$, then

$$f(A \cup B) = f(A) \cup f(B) \tag{2}$$

$$f(A) - f(B) \subset f(A - B). \tag{3}$$

SOLUTION:

1. Let $f : X \to Y$. The image of $A \subset X$, denoted by $f(A)$, is the set of images of points of A

$$f(A) = \{f(x) : x \in A\}. \tag{4}$$

Hence, the image of $[-1, 1]$ is

$$f([-1, 1]) = [0, 1]. \tag{5}$$

2. The inverse image $f^{-1}(B)$ of any subset $B \subset Y$, is the set

$$f^{-1}(B) = \{x : x \in X, f(x) \in B\}. \tag{6}$$

The inverse image of $[1, 4]$ is

$$f^{-1}([1, 4]) = [1, 2] \tag{7}$$

or

$$f^{-1}([1, 4]) = [-1, -2]. \tag{8}$$

3. First, we shall show that

$$f(A \cup B) \subset f(A) \cup f(B). \tag{9}$$

Let $y \in f(A \cup B)$, that is, $x \in A \cup B$ exists such that $f(x) = y$. We have

$$x \in A \quad \text{or} \quad x \in B \quad \text{then}$$

$$x \in A \Rightarrow f(x) = y \in f(A) \text{ or } x \in B \Rightarrow f(x) = y \in f(B). \tag{10}$$

Hence

$$y = f(x) \quad \text{and} \quad y \in f(A) \cup f(B). \tag{11}$$

Now we prove

$$f(A) \cup f(B) \subset f(A \cup B). \tag{12}$$

Let $y \in f(A) \cup f(B)$ then

$$y \in f(A) \quad \text{or} \quad y \in f(B)$$

$$y \in f(A) \Rightarrow \exists x \in A, \text{ such that } f(x) = y$$

$$y \in f(B) \Rightarrow \exists x \in B, \text{ such that } f(x) = y. \tag{13}$$

Hence

$$y = f(x) \quad \text{and} \quad x \in A \cup B, \text{ that is,}$$

$$y \in f(A \cup B).$$

Let $y \in f(A) - f(B)$. Then $\exists x \in A$, such that $f(x) = y$ but $y \notin \{f(x) : x \in B\}$. Hence $x \notin B$ and $x \in A - B$. Then, $y \in f(A - B)$.

● PROBLEM 4–10

Prove the following:
Let $f : X \to Y$, then for any subsets A and B of X,

1. $f(A \cap B) \subset f(A) \cap f(B)$ (1)

2. $A \subset B \Rightarrow f(A) \subset f(B)$ (2)

3. For any family A_α of subsets of X

$$f(\bigcup_\alpha A_\alpha) = \bigcup_\alpha f(A_\alpha) \tag{3}$$

$$f(\bigcap_\alpha A_\alpha) \subset \bigcap_\alpha f(A_\alpha) \tag{4}$$

4. Show that inclusion in (1) cannot be replaced, in general, by equality.

SOLUTION:

1. Suppose $y \in f(A \cap B)$. Hence $x \in A \cap B$ exists, such that $f(x) = y$. But $x \in A$ and $x \in B$, therefore

$$y = f(x) \in f(A) \quad \text{and} \quad y = f(x) \in f(B). \tag{5}$$

Thus

$$y \in f(A) \cap f(B) \tag{6}$$

and

$$f(A \cap B) \subset f(A) \cap f(B). \tag{7}$$

2. Suppose $A \subset B$ and $y \in f(A)$. Then an x exists such that $f(x) = y$ and $x \in$

A. Hence $x \in B$ and $f(x) = y \in f(B)$. Therefore,

$$A \quad B \Rightarrow f(A) \quad f(B). \tag{8}$$

3. In Problem 4-9 we proved that

$$f(A \cup B) = f(A) \cup f(B). \tag{9}$$

Similarly,

$$\begin{aligned} y \in \bigcup_\alpha f(A_\alpha) &\equiv \exists\, \alpha : y \in f(A_\alpha) \equiv \\ \equiv \exists\, x \in A_\alpha : f(x) = y &\equiv \exists\, x \in \bigcup_\alpha A_\alpha : f(x) = y \equiv \\ &\equiv y = f(x) \in f(\bigcup_\alpha A_\alpha). \end{aligned} \tag{10}$$

That proves (3).

Suppose $y \in f(\bigcap_\alpha A_\alpha)$, then

$$\begin{aligned} y \in f(\bigcap_\alpha A_\alpha) &\equiv \exists x \in \bigcap_\alpha A_\alpha : f(x) = y \equiv \\ &\equiv \exists\, x : \forall\, \alpha : x \in A_\alpha : f(x) = y \Rightarrow \\ \exists\, x : \forall\, \alpha &: y \in f(A_\alpha) \equiv y \in \bigcap_\alpha f(A_\alpha). \end{aligned} \tag{11}$$

4. Consider the function $f(x) = x^2$. Let $A = [-1, 0]$ and $B = [0, 1]$ then

$$A \cap B = [0] \quad \text{and} \quad f(A \cap B) = f(0) = [0]$$

$$f(A) = f([-1, 0]) = [0, 1].$$

$$f(B) = [0, 1].$$

$$f(A) \cap f(B) = [0, 1] \neq f(A \cap B) = [0]. \tag{12}$$

● PROBLEM 4–11

Prove that:

$$\begin{pmatrix} f : X \to Y \\ \text{is one} - \text{to} - \text{one} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \forall A, \forall\, B : A, B \subset X \\ f(A \cap B) = f(A) \cap f(B) \end{pmatrix} \tag{1}$$

SOLUTION:

First we shall prove $\Rightarrow$

Suppose $f : X \to Y$ is one-to-one. Let A and B be any subsets of X.

Then, for any $y \in f(A \cap B)$

$$y \in f(A \cap B) \equiv \exists x : x \in A \cap B, f(x) = y \equiv \tag{2}$$

(Note that there is only one $x \in A \cap B$, such that $f(x) = y$.)

$$\equiv x \in A \wedge x \in B \wedge f(x) = y \equiv$$

$$\equiv y \in f(A) \wedge y \in f(B) \equiv y \in f(A \cap B). \tag{3}$$

$\Leftarrow$ Now, suppose for any sets $A \subset X$ and $B \subset X$

$$f(A \cap B) = f(A) \cap f(B) \tag{4}$$

and suppose $f : X \to Y$ is not one-to-one. Then x_1 and x_2, $x_1 \neq x_2$, exist such that

$$y = f(x_1) = f(x_2). \tag{5}$$

Let $A = x_1$ and $B = x_2$; we have

$$f(A) \cap f(B) = f(x_1) \cap f(x_2) = y \neq f(A \cap B) =$$

$$= f(\phi). \tag{6}$$

● PROBLEM 4-12

Let $f : X \to Y$ and let $P(X)$ and $P(Y)$ denote power sets of X and Y respectively. Function f induces functions

$$f : P(X) \to P(Y) \tag{1}$$

by $X \supset A \to f(A)$ and

$$f^{-1} : P(Y) \to P(X) \tag{2}$$

by

$$Y \supset B \to f^{-1}(B).$$

Prove the following:

THEOREM

Let $f : X \to Y$ denote one-to-one. Then the induced function

$$f : P(X) \to P(Y)$$

is also one-to-one.

SOLUTION:

If $X = \phi$, then $P(X) = \{\phi\}$, and

$$f: P(X) \to P(Y)$$

is one-to-one, because no two different elements of $P(X)$ can have the same image, as the set $P(X)$ consists of only one element.

Suppose $X \neq \phi$, then $P(X)$ has at least two elements. Let $A \in P(X)$ and $B \in P(X)$, but $A \neq B$. Then $x \in X$ exists, such that $x \in A$ and $x \notin B$. Hence,

$$f(x) \in f(A) \quad \text{and} \quad f(x) \notin f(B)$$

because f is one-to-one. We find

$$f(A) \neq f(B)$$

therefore, the induced function is also one-to-one.

● PROBLEM 4-13

Prove the following theorem:

THEOREM

Let $f: X \to Y$, then the induced function $f^{-1}: P(Y) \to P(X)$ preserves the elementary set operations

1. $f^{-1}(\bigcup_\alpha B_\alpha) = \bigcup_\alpha f^{-1}(B_\alpha)$ (1)
2. $f^{-1}(\bigcap_\alpha B_\alpha) = \bigcap_\alpha f^{-1}(B_\alpha)$ (2)
3. $f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2)$ (3)

SOLUTION:

1.
$$x \in f^{-1}(\bigcup_\alpha B_\alpha) \equiv f(x) \in \bigcup_\alpha B_\alpha \equiv$$
$$\equiv \exists\, \alpha : f(x) \in B_\alpha \equiv \exists\, \alpha : x \in f^{-1}(B_\alpha) \equiv$$
$$\equiv x \in \bigcup_\alpha f^{-1}(B_\alpha). \tag{4}$$

2.
$$x \in f^{-1}(\bigcap_\alpha B_\alpha) \equiv f(x) \in \bigcap_\alpha B_\alpha \equiv$$
$$\equiv \forall\, \alpha : f(x) \in B_\alpha \equiv \forall\, \alpha : x \in f^{-1}(B_\alpha) \equiv$$
$$\equiv x \in \bigcap_\alpha f^{-1}(B_\alpha). \tag{5}$$

3.
$$x \in f^{-1}(B_1 - B_2) \equiv f(x) \in B_1 - B_2 \equiv$$
$$\equiv f(x) \in B_1 \wedge f(x) \notin B_2 \equiv$$
$$\equiv x \in f^{-1}(B_1) \wedge x \notin f^{-1}(B_2) \equiv$$
$$\equiv x \in f^{-1}(B_1) - f^{-1}(B_2). \tag{6}$$

We conclude that the induced function f^{-1} preserves the elementary set operations. The induced function $f: P(X) \to P(Y)$ preserves unions, but it does not preserve intersections in general.

● PROBLEM 4–14

Show that:
If $f: X \to Y$, then:

1. For each $A \subset X$,
$$A \subset f^{-1}[f(A)]. \tag{1}$$

2. For each $A \subset X$ and $B \subset Y$,
$$B \cap f(A) = f[f^{-1}(B) \cap A] \tag{2}$$

in particular,
$$B \cap f(X) = f[f^{-1}(B)]. \tag{3}$$

SOLUTION:

1. Let A represent any subset of X and $x \in A$. Function f maps x into $f(x) \in Y$. Since function f is not necessarily one-to-one, it is possible that elements $x, x_1, x_2, \ldots$ exist, such that
$$f(x) = f(x_1) = f(x_2) \ldots \text{(See Figure 1).} \tag{4}$$

Therefore,
$$x \in f^{-1}[f(x)] \tag{5}$$

Hence
$$A \subset f^{-1}[f(A)]. \tag{6}$$

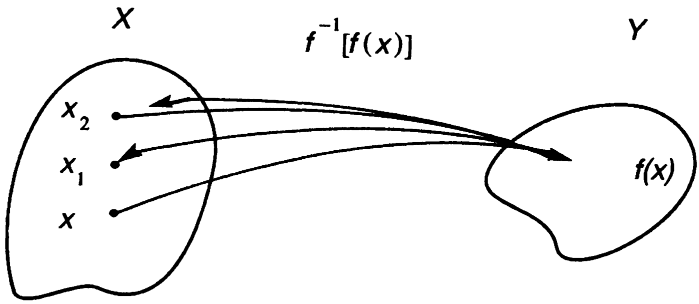


FIGURE 1

2. Suppose $y \in Y$ and

$$y \in f[f^{-1}(B) \cap A] \equiv \exists x : [x \in f^{-1}(B) \cap A] \wedge [f(x) = y] \equiv$$

$$\equiv \exists x : x \in A \wedge x \in f^{-1}(B) \wedge f(x) = y \equiv$$

$$\equiv \exists x : f(x) \in f(A) \wedge f(x) \in B \wedge f(x) = y \equiv$$

$$\equiv y \in f(A) \wedge y \in B \equiv y \in f(A) \cap B. \tag{7}$$

Therefore,

$$f[f^{-1}(B) \cap A] = f(A) \cap B. \tag{8}$$

In particular, setting $A = X$ we obtain

$$f[f^{-1}(B)] = f(X) \cap B \tag{9}$$

because

$$f^{-1}(B) \cap X = f^{-1}(B). \tag{10}$$

● PROBLEM 4–15

Prove this theorem:

THEOREM

Let X represent any set and $\{A_\alpha : \alpha \in \Omega\}$ its covering, i.e.

$$X = \bigcup_\alpha A_\alpha \tag{1}$$

Furthermore, let

$$\forall \alpha \quad \exists f_\alpha : A_\alpha \to Y \tag{2}$$

such that

$$\forall \alpha, \beta \quad f_\alpha \,|\, A_\alpha \cap A_\beta = f_\beta \,|\, A_\alpha \cap A_\beta. \tag{3}$$

Then one, and only one, $f : X \to Y$ exists which is an extension of each f_α, i.e. $\forall \alpha : f \,|\, A_\alpha = f_\alpha$.

SOLUTION:

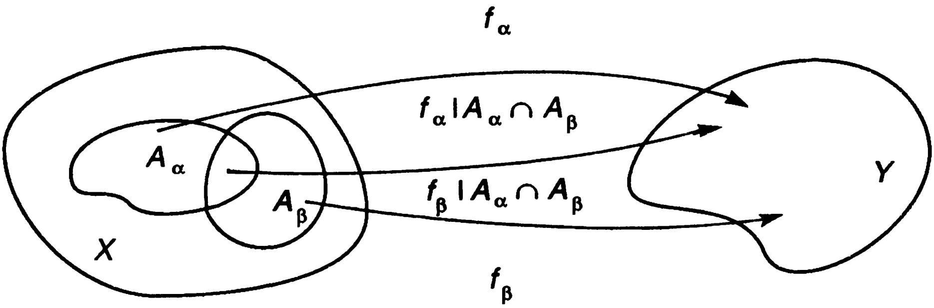


FIGURE 1

Let us define for each $x \in X$, $f(x) = f_\alpha(x)$, where $x \in A_\alpha$. If also $x \in A_\beta$, then $f(x) = f_\beta(x)$ but

$$f_\alpha \,|\, A_\alpha \cap A_\beta(x) = f_\beta \,|\, A_\alpha \cap A_\beta(x). \tag{4}$$

Hence, this is a correct definition of a function: it assigns to an element of X only one element of Y.

$$f(x) = f_\alpha(x) = f_\beta(x). \tag{5}$$

The function

$$f : X \to Y \tag{6}$$

is an extension of each f_α over X. Function f is unique, each x belongs to some A_α (A_α is a covering of X) and the value $f_\alpha(x)$ is uniquely defined.

● PROBLEM 4–16

Let

$$\{A_\alpha : \alpha \in \Omega\}$$

be a covering of X, i.e.

$$X = \bigcup_\alpha A_\alpha$$

such that if $\alpha \neq \beta$, then $A_\alpha \cap A_\beta = \phi$; then the family $\{A_\alpha : \alpha \in \Omega\}$ is called a partition of X.

Show that:
If $\{A_\alpha : \alpha \in \Omega\}$ is a partition of X and for each A_α, f_α is given, such that $f_\alpha : A_\alpha \to Y$, then $f : X \to Y$, which is an extension of each f_α, is uniquely defined by

$$f(x) = f_\alpha(x) \quad \text{for} \quad x \in A_\alpha. \tag{1}$$

SOLUTION:

We shall use the theorem proved in Problem 4-15. Since $\{A_\alpha : \alpha \in \Omega\}$ is a partition of X, the condition

$$f_\alpha \,|\, A_\alpha \cap A_\beta = f_\beta \,|\, A_\alpha \cap A_\beta \tag{2}$$

is satisfied. Therefore, (1) defines a function on X which is unique.

● PROBLEM 4-17

Let

$$\{A_\alpha : \alpha \in \Omega\} \text{ and } \{B_\beta : \beta \in \Lambda\} \tag{1}$$

denote two coverings (partitions) of X. Show that

$$\{A_\alpha \cap B_\beta : (\alpha, \beta) \in \Omega \times \Lambda\} \tag{2}$$

is also a covering (partition) of X.

SOLUTION:

Both $\{A_\alpha\}$ and $\{B_\beta\}$ are coverings of X. Let $x \in X$ represent any element of X. Since

$$X = \bigcup_\alpha A_\alpha$$

α' exists, such that $x \in A_{\alpha'}$. Also, β' exists, such that $x \in A_{\beta'}$. Therefore, $x \in A_{\alpha'} \cap B_{\beta'}$.

Hence,

$$\bigcup_{(\alpha,\beta)} A_\alpha \cap A_\beta = X \tag{3}$$

and (2) is a covering of X. Now we shall show that if both $\{A_\alpha\}$ and $\{B_\beta\}$ are partitions, then $\{A_\alpha \cap B_\beta\}$ is also a partition.

Consider,

$$(A_\alpha \cap B_\beta) \cap (A_{\alpha'} \cap B_{\beta'}) =$$

$$= (A_\alpha \cap A_{\alpha'}) \cap (B_\beta \cap B_{\beta'}). \tag{4}$$

Then

$$A_\alpha \cap A_{\alpha'} = \phi \quad \text{for } \alpha \neq \alpha'$$

and

$$B_\beta \cap B_{\beta'} = \phi \quad \text{for } \beta \neq \beta'.$$

Therefore, $\{A_\alpha \cap B_\beta\}$ is a partition.

● PROBLEM 4–18

Let $\{A_\alpha : \alpha \in \Omega\}$ and $\{B_\beta : \beta \in \Lambda\}$ denote coverings (partitions) of sets X and Y, respectively. Show that

$$\{A_\alpha \times B_\beta : (\alpha, \beta) \in \Omega \times \Lambda\} \tag{1}$$

is a covering (partition) of $X \times Y$.

SOLUTION:

Let $(x, y) \in X \times Y$. Then $x \in X$ and $y \in Y$. $\{A_\alpha\}$ is a covering of X. Hence

$$X = \bigcup_\alpha A_\alpha \tag{2}$$

and

$$\exists \alpha' \in \Omega : x \in A_{\alpha'}. \tag{3}$$

$\{B_\beta\}$ is a covering of Y. Hence

$$Y = \bigcup_\beta B_\beta \tag{4}$$

and

$$\exists \beta' \in \Lambda : y \in B_{\beta'}. \tag{5}$$

Thus,

$$(x, y) \in A_{\alpha'} \times B_{\beta'} \tag{6}$$

and (1) is a covering of $X \times Y$

$$X \times Y = \bigcup_{(\alpha,\beta)} A_\alpha \times B_\beta. \tag{7}$$

Suppose $\{A_\alpha\}$ and $\{B_\beta\}$ are partitions of X and Y, respectively; then

$$(A_\alpha \times B_\beta) \cap (A_{\alpha'} \times B_{\beta'}) = (A_\alpha \cap A_{\alpha'}) \times (B_\beta \cap B_{\beta'}) =$$

$$= \phi \times \phi = \phi. \tag{8}$$

Hence, $\{A_\alpha \times B_\beta\}$ is also a partition.

Let T be a family of subsets of X. Show that the smallest additive family U exists, such that $T \subset U$.

SOLUTION:

DEFINITION (ADDITIVE FAMILY)

The family of sets U is additive if

$$(A \in U, B \in U) \Rightarrow (A \cup B \in U) \tag{1}$$

Let N represent the class of all additive families S, such that

$$T \subset S. \tag{2}$$

Since the family of all subsets of X is an element of N, $N \neq \phi$. Let

$$U = \cap N. \tag{3}$$

We shall show that U defined by (3) is additive.

Let $A \in U$ and $B \in U$, then for every $S \in N$, $A \in S$ and $B \in S$. The set N consists of additive families, therefore,

$$A \cup B \in S. \tag{4}$$

Equation (4) holds for every $S \in N$, hence

$$A \cup B \in U. \tag{5}$$

U is additive.

Now, we shall show $T \subset U$. By assumption:

$$\text{for every } S \in N,\ T \subset S.$$

Hence, if $A \in T$, then $A \in S$ and

$$A \in U. \tag{6}$$

Therefore,

$$A \in T \Rightarrow A \in U \tag{7}$$

or

$$T \subset U.$$

The family $U = \cap N$ is the smallest additive family containing the family T, because U is the intersection of all families with this property.

We define:

The family S of sets is multiplicative, if

$$(A \in S \wedge B \in S) \Rightarrow (A \cap B \in S). \qquad (8)$$

For multiplicative families, a similar theorem exists.

THEOREM

For every family T of subsets of X, the smallest multiplicative family U exists, such that $T \subset U$.

● PROBLEM 4–20

Consider the set X and the family T of all its one-element subsets. Find the smallest additive family U, such that $T \subset U$.

SOLUTION:

Since T is the family of all one-element subsets of X, we have

$$\forall\{a\}\ \forall\{b\},\ \{a\} \in T,\ \ \{b\} \in T$$

$$\{a\} \cup \{b\} = \{a, b\} \in U \qquad (1)$$

Because U is additive and $T \subset U$, U contains all one- and two-element subsets. By the same token, we can show that U consists of all finite subsets of X.

From the above considerations, it follows that a necessary and sufficient condition for a set X to be finite is that the family of all its non-empty subsets must be identical with U.

Note that by using this property, we can define a finite set without referring to the concept of natural numbers.

● PROBLEM 4–21

Let S denote a family of sets and S' denote the family of all sets of the form $D' = A - B$, where $A, B \in S$. Prove that

$$S' \subset S''. \qquad (1)$$

SOLUTION:

The set S' consists of

$$S' = \{D' : D' = A - B, A \in S \wedge B \in S\} \qquad (2)$$

while S'' consists of

$$S'' = \{D'' : D'' = E' - F', E' \in S' \wedge F' \in S'\}. \tag{3}$$

We understand that

$$S'' = (S')' \tag{4}$$

From (2) and (3), we obtain

$$S'' = \{D'' : D'' = (A - B) - (G - H), A, B, G, H \in S\} \tag{5}$$

where $E' = A - B$ and $F' = G - H$.

Suppose $A' \in S'$, then

$$A' = A - B \tag{6}$$

where $A, B \in S$.

Since S' consists of all sets of the form $A - B$, we conclude that

$$\phi \in S' \tag{7}$$

Therefore,

$$(A - B) - (A - A) = (A - B) - \phi = A - B \in S'' \tag{8}$$

and

$$S' \subset S''. \tag{9}$$

The inverse inclusion, in general, is false.

● PROBLEM 4–22

Prove that:

$$(A^B = \phi) \Leftrightarrow (A = \phi \wedge B \neq \phi). \tag{1}$$

Remember that $A^B = \{f : f : B \to A\}$.

SOLUTION:

$\Leftarrow$ Suppose $A = \phi$ and $B \neq \phi$. Then $f \in A^B$ implies $f \subset B \times A = B \times \phi = \phi$. But ϕ is not a function $f : B \to A, B \neq \phi$ and $A = \phi$. Therefore,

$$A^B = \phi. \tag{2}$$

$\Rightarrow$ Suppose $A \neq \phi$. Then $\exists a : a \in A$ and

$$\{(x, a) : x \in B\}$$

is a function from B to A. Hence

$$A^B \neq \phi. \tag{3}$$

Now, suppose $B = \phi$, then ϕ is a function from B to A and

$$A^B \neq \phi.$$

● PROBLEM 4–23

Prove that:

$$(A^B = B^A) \Rightarrow (A = B). \qquad (1)$$

SOLUTION:

Suppose $A^B \neq \phi$, then a function f exists, such that

$$f : B \to A$$

$F \in A^B = B^A$. Since $f \in A^B$, the domain of f is B.

On the other hand, since $F \in B^A$, the domain of f is A. Thus,

$$A = B. \qquad (2)$$

Now, suppose $A^B = \phi$ then

$$(A^B = \phi) \Rightarrow (A = \phi \wedge B \neq \phi). \qquad (3)$$

Hence $A = \phi$.

But $A^B = B^A = \phi$ and $B = \phi$. Therefore,

$$A = B. \qquad (4)$$

● PROBLEM 4–24

Let $F(X, R)$ denote the set of all real-valued functions $f \in F(X, R)$

$$f : X \to R. \qquad (1)$$

Show that $F(X, R)$, with the usual algebraic operations, forms a real linear vector space.

SOLUTION:

We shall define algebraic operations on the elements $F(X, R)$. Let $f : X \to R$ and $g : X \to R$ and $a \in R$, then we define

$$(f + g)(x) = f(x) + g(x) \qquad (2)$$

$$(a \cdot f)(x) = a \cdot f(x) \qquad (3)$$

$$(f \cdot g)(x) = f(x) \cdot g(x) \tag{4}$$

$$(f + a)(x) = f(x) + a \tag{5}$$

We shall verify that $F(X, R)$ satisfies the axioms of the real linear vector space:

1. $(f + g) + (h) = f + (g + h)$
2. $f + g = g + f$
3. $\exists 0 \in F(X, R)$, such that $\forall f \in F(X, R), f + 0 = f$
4. $\forall f \in F(X, R) \quad \exists -f \in F(X, R)$, such that $f + (-f) = 0$
5. $a \cdot (b \cdot f) = (ab) \cdot f$ for any real numbers a and b
6. $1 \cdot f = f$
7. $a \cdot (f + g) = af + ag$
8. $(a + b) f = af + bf$

● PROBLEM 4–25

Let X represent the space (universal set), and A any subset of X. The characteristic function of A is defined by

$$X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{1}$$

Show that

1. $X_{A \cap B} = X_A X_B \qquad (2)$
2. $X_{A \cup B} = X_A + X_B - X_{A \cap B} \qquad (3)$
3. $X_{A - B} = X_A - X_{A \cap B} \qquad (4)$

SOLUTION:

1. We shall prove that

$$X_{A \cap B} = X_A \cdot X_B \tag{5}$$

Suppose $x \in A \cap B$, then

$$X_{A \cap B} = 1 \text{ and } X_A = X_B = 1.$$

If, on the other hand, $x \notin A \cap B$, then $X_{A \cap B} = 0$ and at least one of the functions X_A or X_B is equal to zero, $X_A \cdot X_B = 0$.

2. $X_{A \cup B} = 1$ when

   (a) $x \in A \wedge x \notin B$

   (b) $x \notin A \wedge x \in B$

   (c) $x \in A \wedge x \in B$

For (a) we have $X_A + X_B - X_{A \cap B} = 1 + 0 - 0 = 1$; for (b) we have $X_A + X_B - X_{A \cap B} = 0 + 1 - 0 = 1$; for (c) we have $X_A + X_B - X_{A \cap B} = 1 + 1 - 1 = 1$.

If $X_{A \cup B} = 0$, then $x \notin A$ and $x \notin B$ hence $X_A + X_B - X_{A \cap B} = 0$.

3. Suppose $X_{A-B} = 1$, then $x \in A$ and $x \notin B$,

$$X_A - X_{A \cap B} = 1 - 0 = 1$$

If $X_{A-B} = 0$, then

   (a) $x \notin A$, or

   (b) $x \in A \wedge x \in B$

For (a) $X_A - X_{A \cap B} = 0 - 0 = 0$; for (b) $X_A - X_{A \cap B} = 1 - 1 = 0$.

● PROBLEM 4-26

Prove the following:

THEOREM

Let $f : A \to B$ represent relation-preserving, and let R represent relation in A, and S relation in B. Then, there is one, and only one function

such that $$F : \frac{A}{R} \to \frac{B}{S}, \tag{1}$$

$$p_B \circ f = F \circ p_A \tag{2}$$

F is called the function induced by f and $p_A : A \to {}^{A}/_{R}$ and $p_B : B \to {}^{B}/_{R}$.

SOLUTION:

Let us start with the definition.

DEFINITION (RELATION-PRESERVING FUNCTION)

Let A, B denote two sets with equivalence relations R and S respectively. A function $f : A \to B$ is called relation-preserving if

$$(a, b) \in R \Rightarrow [f(a), f(b)] \in S. \tag{3}$$

Now we shall show that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \\ \frac{A}{R} \arrow[ur, "F"'] & \end{tikzcd}$$

FIGURE 1

commutes. An equivalence class (see Problem 3–23) is denoted by $[a]$. Let us define function F by

$$F : [a] \to [f(a)] \tag{4}$$

where $[a] \in {}^{A}/_{R}$ and $[f(a)] \in {}^{B}/_{S}$. Note that the equivalence class, $[f(a)] \in {}^{B}/_{S}$ is independent of the representative $a \in [a]$.

Take for example, $a' \in [a]$, then $(a, a') \in R$; but since f is relation preserving,

$$(a, a') \in R \Rightarrow [f(a), f(a')] \in S. \tag{5}$$

Therefore,

$$f(a) \in [f(a)] \quad \text{and} \quad f(a') \in [f(a)]$$

and F defined by (4) is uniquely defined.

Function F is unique because p_A is a surjective. Suppose F^* is another function for which the diagram commutes, then for at least one $[a] \in {}^{A}/_{R}$,

$$F^*([a]) \neq f([a]). \tag{6}$$

But since p_A is a surjective, (6) leads to

$$F^* \circ p_A(a) \neq F \circ p_A(a) \tag{7}$$

which is impossible, because of commutativity.

That the diagram commutes follows from

$$p_B \circ f(a) = [f(a)] = F([a]) = F \circ p_A(a). \tag{8}$$

● PROBLEM 4–27

Let $f : A \to B$. Show that

$$(a, b) \in R \Leftrightarrow f(a) = f(b) \tag{1}$$

is an equivalence relation and show that a function $F : {}^{A}/_{R} \to B$ exists, such that

$$f = F \circ p \tag{2}$$

i.e. show that the diagram (Figure 1) commutes.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p_a"'] & B \arrow[d, "p_b"] \\
\frac{A}{R} \arrow[r, "F"] & \frac{B}{S}
\end{tikzcd}$$

FIGURE 1

SOLUTION:

We shall show that (1) defines an equivalence relation.

1. $\forall a \in A : f(a) = f(a) \Rightarrow (a, a) \in R.$

2. $\forall a, b \in A : \ (a, b) \in R \Leftrightarrow f(a) = f(b) \Leftrightarrow (b, a) \in R.$

3. $\forall a, b, c \in A : (a, b) \in R \wedge (b, c) \in R \Leftrightarrow f(a) = f(b) \wedge f(b) = f(c) \Rightarrow$
$\Rightarrow f(a) = f(c) \Leftrightarrow (a, c) \in R$

Hence, R is an equivalence relation.

Let us define $F : {}^{A}/_{R} \to B$ by

$$F([a]) = f(a). \tag{3}$$

Function F is uniquely defined. Indeed, suppose $b \in [a]$, then

$$(a, b) \in R \Rightarrow f(a) = f(b). \tag{4}$$

The diagram commutes because

$$(F \circ p)(a) = F(p(a)) = F([a]) = f(a). \tag{5}$$

● PROBLEM 4-28

Let $\{A_\alpha : \alpha \in \Omega\}$ be a family of sets. Define the Cartesian product of $\{A_\alpha\}$.

SOLUTION:

The Cartesian product of two sets A_1 and A_2, denoted $A_1 \times A_2$, was defined as a set of all ordered pairs

$$A_1 \times A_2 = \{(a, b) : a \in A_1 \wedge b \in A_2\} \tag{1}$$

On the other hand, $A_1 \times A_2$ can be considered to be the set of functions f of the index set $\{1, 2\}$ into $A_1 \cup A_2$, such that

$$f : \{1, 2\} \to A_1 \cup A_2$$

$$f(1) \in A_1 \quad \text{and} \quad f(2) \in A_2. \tag{2}$$

DEFINITION

Let $\{A_\alpha : \alpha \in \Omega\}$ represent the family of sets. The Cartesian product, denoted $\underset{\alpha}{X} A_\alpha$, is the set of all maps

$$F : \Omega \to \bigcup_\alpha A_\alpha \tag{3}$$

such that

$$\forall \alpha \in \Omega : F(\alpha) \in A_\alpha.$$

An element $f \in \underset{\alpha}{X} A_\alpha$ is usually denoted as $\{a_\alpha\}$, where $\forall \alpha \in \Omega : f(\alpha) \in a_\alpha$.

The element $a_\alpha \in A_\alpha$ is called the α^{th} coordinate of $\{a_\alpha\}$.

The set A_α is called the α^{th} factor of $\underset{\alpha}{X} A_\alpha$.

When the index set of Ω is the set of natural numbers and all A_n are $A_n = R$, where R is the set of real numbers, we obtain

$$\underset{m=1}{\overset{n}{X}} R = R^n$$

the n-dimensional Euclidean space; and the product,

$$\overset{\infty}{\underset{n=1}{X}} R_n$$

is the extension of the n-dimensional space R^n to an infinite number of dimensions.

● PROBLEM 4-29

Prove that the following properties are equivalent:

1. If each $A_\alpha \neq \phi$ and $\{A_\alpha : \alpha \in \Omega\}$ is a non-empty family, then $\underset{\alpha}{X} A_\alpha \neq \phi$
2. The axiom of choice.
3. If each $A_\alpha \neq \phi$ and $\{A_\alpha : \alpha \in \Omega\}$ is a non-empty family of sets, then a function $F : \Omega \to \bigcup_\alpha A_\alpha$ exists such that $\forall \alpha \in \Omega : F(\alpha) \in A_\alpha$. F is called the choice function.

SOLUTION:

(1) ⇒ (2).

Suppose $\{A_\alpha : \alpha \in \Omega\}$ is a non-empty family of sets. Since $\underset{\alpha}{X} A_\alpha \neq \phi$, an element $F = \{a_\alpha\}$ exists and

$$F(\Omega)$$

is a set which satisfies the axiom of choice.

(2) ⇒ (3).

Let us define for each $\alpha \in \Omega$

$$A'_\alpha = \{\alpha\} \times A_\alpha$$

Each A'_α is a non-empty set.

According to the axiom of choice, a set T exists consisting of exactly one element from each A'_α. Hence, for each α, there is a unique $(\alpha, a_\alpha) \in T$, $a_\alpha \in A_\alpha$

$$T \subset \bigcup_\alpha(\{\alpha\} \times A_\alpha) \subset \bigcup_\alpha(\Omega \times A_\alpha) = \Omega \times \bigcup_\alpha A_\alpha$$

and T is a function

$$T : \Omega \to \bigcup_\alpha A_\alpha .$$

(3) ⇒ (1).

If $F : \Omega \to \bigcup_\alpha A_\alpha$, then F is an element of $\underset{\alpha}{X} A_\alpha$.

Let $\{A_\alpha : \alpha \in \Omega\}$ represent a family of non-empty sets, and let

$$\Lambda \subset \Omega.$$

Show that the function

$$P : \underset{\alpha}{X}\{A_\alpha : \alpha \in \Omega\} \rightarrow \underset{\alpha}{X}\{A_\alpha : \alpha \in \Lambda\} \tag{1}$$

defined by

$$P(F) = F \mid \Lambda \tag{2}$$

is onto and that each projection

$$p_\beta : \underset{\alpha}{X} A_\alpha \rightarrow A_\beta \tag{3}$$

is onto.

SOLUTION:

Let f be any element

$$f \in X\{A_\alpha : \alpha \in \Lambda\}. \tag{4}$$

We shall show that

$$F \in X\{A_\alpha : \alpha \in \Omega\} \tag{5}$$

exists, such that

$$P(F) = f. \tag{6}$$

By Problem 4-29 part 3, a choice function exists such that

$$G : \Omega - \Lambda \rightarrow \cup\{A_\alpha : \alpha \in \Omega - \Lambda\} \tag{7}$$

Then

$$F : \Omega \rightarrow \cup\{A_\alpha : \alpha \in \Omega\} \tag{8}$$

defined by

$$F \mid \Lambda = f$$

$$F \mid \Omega - \Lambda = G \tag{9}$$

is an element

$$F \in X\{A_\alpha : \alpha \in \Omega\} \tag{10}$$

and

$$P(F) = F \mid \Lambda = f.$$

Setting $\Lambda = \{\beta\}$, $\beta \in \Omega$ we obtain $p_B = P$, which completes the proof.

Two families are given $\{A_\alpha : \alpha \in \Omega\}$ and $\{B_\beta : \beta \in \Lambda\}$. Prove that

$$\left(\bigcup_{\alpha\in\Omega} A_\alpha\right) \times \left(\bigcup_{\beta\in\Lambda} B_\beta\right) = \bigcup_{(\alpha,\beta)\in\Omega\times\Lambda} (A_\alpha \times B_\beta). \tag{1}$$

SOLUTION:

Suppose

$$(a,b) \in (\bigcup_\alpha A_\alpha) \times (\bigcup_\beta B_\beta) \tag{2}$$

then

$$a \in \bigcup_{\alpha\in\Omega} A_\alpha \text{ and } b \in \bigcup_{\beta\in\Lambda} B_\beta. \tag{3}$$

Hence, for some $\alpha \in \Omega$, $a \in A_\alpha$ and for some $\beta \in \Lambda$, $b \in B_\beta$. Therefore,

$$a \in A_\alpha \text{ and } b \in B_\beta \text{ for some } (\alpha, \beta) \in \Omega \times \Lambda.$$

Therefore,

$$(a, b) \in A_\alpha \times B_\beta \text{ for some } (\alpha, \beta) \in \Omega \times \Lambda \tag{4}$$

and finally

$$(a,b) \in \bigcup_{(\alpha,\beta)\in\Omega\times\Lambda} A_\alpha \times B_\beta. \tag{5}$$

CHAPTER 5

POWER OF A SET, EQUIVALENT SETS

● PROBLEM 5–1

Let $N = \{1, 2, 3, ...\}$ and $P = \{2, 4, 6, ...\}$. Show that N and P are equivalent sets. Are they infinite?

SOLUTION:

We shall start with a very important definition.

DEFINITION OF EQUIVALENT SETS

Two sets A and B are called equivalent, written $A \sim B$, if a function $f : A \to B$ exists, which is one-to-one and onto. Function f defines a one-to-one correspondence between the sets A and B.

Let us define

$$f : n \to 2n, \quad f : N \to P. \tag{1}$$

Function (1) is one-to-one and onto. Therefore, sets N and P are equivalent.
Here is another definition.

DEFINITION OF FINITE SETS

A set is finite iff it is empty or equivalent to $\{1, 2, ..., n\}$ for some $n \in N$. Otherwise, the set is called infinite.

The set N is infinite because it is not empty and not equivalent to any set $\{1, 2, ..., n\}$. Since N and P are equivalent, P is also infinite.

● PROBLEM 5–2

Determine which of the following pairs are equivalent sets.

1. $\{1, 2, 3\}$ and $\{a, b, 4\}$.

2. Points of two circles.

SOLUTION:

1. There is a one-to-one and onto correspondence between the sets $\{1, 2, 3\}$ and $\{a, b, 4\}$.

$$\begin{aligned} f : 1 &\to a \\ 2 &\to b \end{aligned}$$

$3 \to 4$

Two finite sets are equivalent if, and only if, they contain the same number of elements.

2. We can assume that circles are concentric.

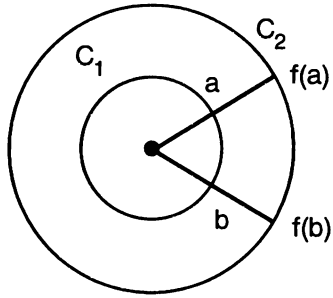


FIGURE 1

If they have equal radii, then equivalence is obvious. If $r_1 \neq r_2$, we can establish a one-to-one and onto correspondence in the following way:

Let a represent any point on the circle C_1. Then $f(a)$ is the point of intersection of the radius from the center, through a, to C_2, as shown in Figure 1. Function f defined above is one-to-one and onto. Hence the sets consisting of the points of C_1 and C_2 are equivalent.

● PROBLEM 5-3

Show that the open interval $]-1, 1[$ and the real axis R are equivalent.

SOLUTION:

Consider the "structure" shown in Figure 1.

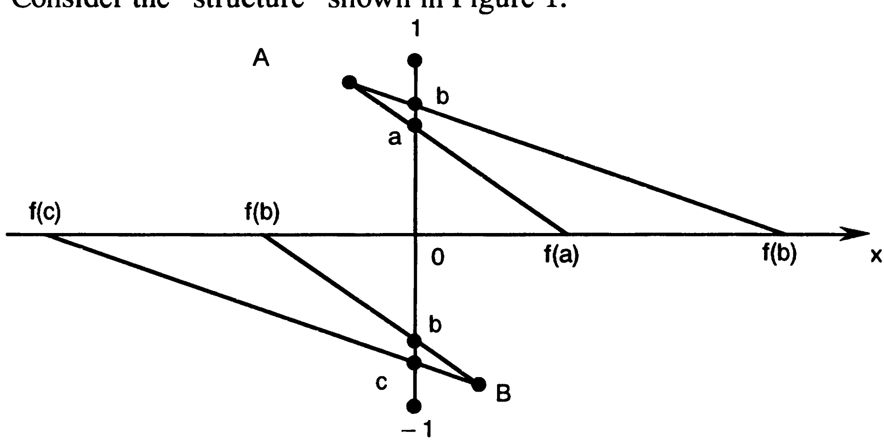


FIGURE 1

The "upper" half of the interval is mapped on the positive part of the real axis in the following manner:

Let a represent any point, such that $a \in]1, 0]$. We will draw a line from point A (which is at the same level as point 1 of the interval, and to the left of it) through point a to the x-axis. The point of intersection is $f(a)$. In the same manner, we assign a unique point of the negative axis ($x < 0$) to each point of $]0, -1[$. The determined function is one-to-one and onto. Hence, the open interval $]-1, 1[$ is equivalent to R.

Another way to show it is to define function f, such that

$$f:]-1, 1[\rightarrow R \text{ and } f(x) = \frac{x}{1 - |x|}.$$

Function f is onto and one-to-one. The desired result follows.

● PROBLEM 5-4

1. Show that the set of elements of any infinite sequence of distinct terms is denumerable.

2. Prove that the relation $A \sim B$ (A is equivalent to B) is an equivalence relation.

SOLUTION:

1. We shall start with a definition.

DEFINITION OF DENUMERABLE SETS

Set A is called denumerable if it is equivalent to the set of positive integers $N = \{1, 2, 3, \ldots\}$. Such a set is said to have cardinality aleph-null, denoted $\aleph_0$ or card N.

Let us define the function $f: N \rightarrow \{a_n\}$ by

$$f: n \rightarrow a_n.$$

Its domain is the set N and its range is the set of all elements of the sequence (a_n).

Since the elements of (a_n) are distinct, the function is one-to-one and onto. Hence, the set of elements of an infinite sequence of distinct terms is denumerable.

DEFINITION OF COUNTABLE SETS

A set is called countable if it is finite or denumerable.

Any sequence is a countable set.

2. We shall show that $A \sim B$ is an equivalence relation. The identity function , i.e.

$$1 : x \to x$$

is a one-to-one mapping of the set A onto itself. Since the inverse of a one-to-one mapping is one-to-one, it follows that

$$(A \sim B) \Rightarrow (B \sim A).$$

The composition of two one-to-one and onto mappings is one-to-one and onto. Therefore,

$$(A \sim B) \wedge (B \sim C) \Rightarrow A \sim C.$$

Remark:

Relation $\sim$ is an equivalence relation. Hence we can classify the sets with respect to their power.

To each set A we assign a cardinal number (denoted sometimes by $\overline{\overline{A}}$ or card A) in such a way that the same cardinal number is assigned to two distinct sets if these sets are equivalent (have the same power).

● PROBLEM 5-5

1. Show that the lattice points (n, m) in a plane form a denumerable set, where n and m are integers.

2. Show that the set of positive rational numbers is denumerable.

SOLUTION:

1. Let us order the lattice points in the manner shown in Figure 1.

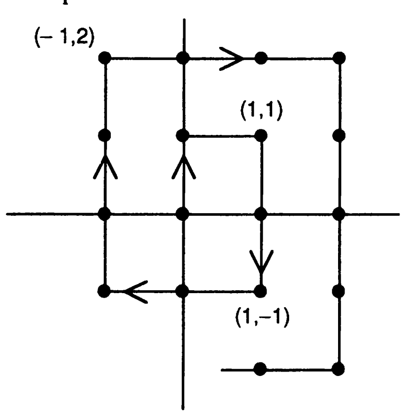


FIGURE 1

We order the set of lattice points into a sequence. That is, we establish a one-to-one and onto correspondence between the set of natural numbers, and the set of lattice points.

The lattice points are ordered as follows:

(0,0) – 1
(0,1) – 2
(1,1) – 3
(1,0) – 4
(1, –1) – 5 etc.

The ordering starts at the origin.

2. The set of rational numbers is arranged as shown in Figure 2.

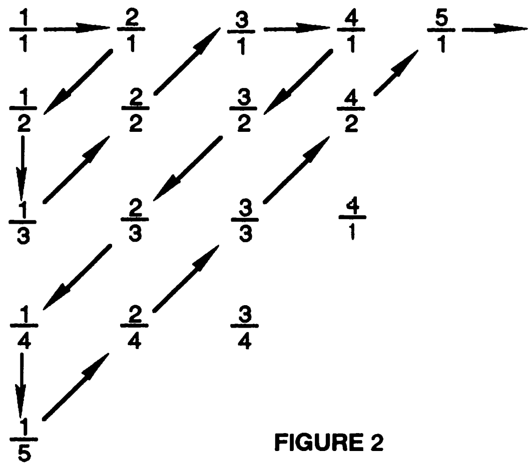


FIGURE 2

The diagram is traversed in the diagonally arrowed path. Every positive rational number will be included. This procedure establishes a one-to-one and onto mapping between the set of positive integers N and the set of rational positive numbers

1 – $^1/_1$
2 – $^2/_1$
3 – $^1/_2$
4 – $^1/_3$
5 – $^2/_3$
⋮

Hence, the set of rational numbers is denumerable.

● PROBLEM 5-6

Prove the following:

THEOREM

The set of all real numbers is non-countable.

SOLUTION:

To prove this theorem, it suffices to prove that for every sequence of real numbers $a_1, a_2, \ldots, a_n, \ldots$ a real number, a, which does not belong to the sequence, exists.

We define a sequence of closed intervals

$$\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_n\beta_n, \ldots \tag{1}$$

such that

$$\beta_n - \alpha_n = \frac{1}{3^n} \quad \text{and} \quad \alpha_n\beta_n \subset \alpha_{n-1}\beta_{n-1},\ a_n \notin \alpha_n\beta_n \tag{2}$$

We divide the closed interval $[0, 1]$ into three $[0, 1/3]$, $[1/3, 2/3]$, $[2/3, 1]$ and choose one which does not contain the point a_1. That interval is $\alpha_1\beta_1$. Similarly, we divide $\alpha_1\beta_1$ into three intervals and choose the one which does not contain a_2. That interval is $\alpha_2\beta_2$. So, in the closed interval $\alpha_{n-1}\beta_{n-1}$ we determine a closed interval $\alpha_n\beta_n$ of length $1/3^n$, which does not contain the point a_n. Let a denote the common point of all the closed intervals $\alpha_n\beta_n$

$$a = \bigcap_{n=1}^{\infty} \alpha_n\beta_n \tag{3}$$

Then

$$a = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n. \tag{4}$$

For every n, $a \in \alpha_n\beta_n$ while $a \neq a_n$, because $a_n \notin \alpha_n\beta_n$.

We have a sequence of real numbers $(a_1, a_2, a_3, \ldots)$. A real number, a, exists, which does not belong to the sequence.

Hence, the set of real numbers is not countable.

• PROBLEM 5-7

Show that the union $A \cup B$ of two countable sets A and B is countable.

SOLUTION:

The set A is countable, therefore its elements can be written in the form of an infinite sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots \tag{1}$$

Similarly, the elements of B can be written in the form of a sequence

$$b_1, b_2, \ldots, b_n, \ldots \tag{2}$$

We form the sequence

$$a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_n, b_n, \ldots \tag{3}$$

The terms of sequence (3) form the set $A \cup B$. Hence $A \cup B$ is a countable set.

From this, we conclude that the set of all integers is countable.

The set of all positive integers is countable and the set of all negative integers is countable.

● PROBLEM 5-8

Prove that the set of all rational numbers is denumerable.

SOLUTION:

Let Q^+ denote the set of positive rational numbers and let Q^- be the set of negative rational numbers. Then the set of all rational numbers Q is

$$Q = Q^+ \cup Q^- \cup \{0\}. \tag{1}$$

In Problem 5-5 we proved that the set of positive rational numbers Q^+ is denumerable. By the same token, we can show that the set of negative rational numbers Q^- is denumerable. The sets Q^+, Q^- and $\{0\}$ are countable.

From Problem 5-7, we conclude that the set Q is denumerable as the union of Q^+, Q^- and $\{0\}$.

● PROBLEM 5-9

Prove the following:

THEOREM

The Cartesian product of two (or, more generally, of a finite number of) countable sets is a countable set.

SOLUTION:

First we shall prove that the set of pairs (m, n), where m and n are natural numbers, is countable.

We will arrange the pairs in a sequence, in such a way that if

$$m + n < m' + n' \tag{1}$$

then (m, n) comes before (m', n'). If $m + n = m' + n'$, then the pair with the smaller antecedent comes first. We obtain the sequence of pairs

$$(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), (2, 3), \ldots \tag{2}$$

Similarly, using the indexes of a_n and b_m, we write

$$(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_1, b_3), (a_2, b_2), (a_3, b_1), (a_1, b_4), \dots \quad (3)$$

Suppose A and B are countable sets, then

$$A = \{a_1, a_2, a_3, a_4, \dots\} \text{ and } B = \{b_1, b_2, b_3, \dots\} \quad (4)$$

The Cartesian product of A and B is

$$A \times B = \{(a_n, b_m) : a_n \in A \wedge b_m \in B\}. \quad (5)$$

The elements of $A \times B$ can be arranged in a sequence as shown in (3).

If the sets A, B, and C are countable, then $A \times B$ is countable, and

$$(A \times B) \times C = A \times B \times C$$

is a countable set. Hence, the Cartesian product of a finite number of countable sets is a countable set.

It is easy to conclude from the above theorem that the set of all rational numbers is countable.

● PROBLEM 5-10

Prove:
The collection P of all polynomials with integer coefficients

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \quad (1)$$

is a denumerable set.

SOLUTION:

Let us define for each pair

$$(m, n) \in N \times N \quad (2)$$

the set P_{mn} of polynomials $p(x)$ of degree n, such that

$$|a_0| + |a_1| + \dots + |a_n| = m. \quad (3)$$

For each pair (m, n), the set P_{mn} is finite.

The set of all polynomials P can be represented as

$$P = \bigcup_{m,n} \{P_{mn} : (m, n) \in N \times N\}. \quad (4)$$

The set P is a countable union of countable sets, therefore, P is a countable set. Since P is not finite, P is denumerable.

• PROBLEM 5-11

Prove that the set Ω of algebraic numbers is denumerable.

SOLUTION:

A real number q is called an algebraic number if q is a solution to a polynomial equation

$$a_m x^m + \ldots + a_1 x + a_0 = 0 \tag{1}$$

with integral coefficients.

The set P of all polynomials with integer coefficients is a denumerable set (see Problem 5-10). Hence

$$P = \{f_1(x) = 0, f_2(x) = 0, f_3(x) = 0, \ldots\} \tag{2}$$

where $f_1, f_2, f_3, \ldots$ are polynomials.

Let us denote by Q_i

$$Q_i = \{x : f_i(x) = 0\} \tag{3}$$

the set of the solutions of $f_i(x) = 0$. Since a polynomial of degree m has, at most, m solutions, each Q_i is finite. Therefore

$$\Omega = \bigcup_{i \in N} Q_i \tag{4}$$

is denumerable.

The set of all real numbers is non-countable. Each real number is either algebraic or transcendental (i.e. non-algebraic),

$$R = \Omega \cup T$$

where T is the set of transcendental numbers. We conclude that transcendental numbers exist and that the set T of transcendental numbers is non-countable. The members e and π are transcendental numbers.

• PROBLEM 5-12

Use the axiom of choice to prove:

THEOREM

Every infinite set A contains a subset B which is denumerable.

SOLUTION:

Let $P(A)$ represent the power set of A, that is, the class of all subsets of A. We define the choice function F, such that

$$F : P(A) \to A. \tag{1}$$

For each non-empty subset D of A, $D \subset A$, $f(D) \in A$. By virtue of the axiom of choice, such a function exists. We shall start from the "top" and move "down":

$$F(A) = a_1$$

$$F(A - \{a_1\}) = a_2$$

$$F(A - \{a_1, a_2\}) = a_3$$

$$\vdots$$

$$F(A - \{a_1, a_2, \dots, a_n\}) = a_{n+1}. \tag{2}$$

The set A is infinite, hence for every $n \in N$, the set

$$A - \{a_1, a_2, \dots, a_n\} \neq \phi \tag{3}$$

is non-empty.

Since F is a choice function

$$a_n \neq a_k \quad \text{for} \quad n \neq k \tag{4}$$

Hence all a_n are distinct and the set

$$B = \{a_1, a_2, a_3, \dots\} \tag{5}$$

is a denumerable subset of A, $B \subset A$. In the first step, the choice function F chooses one element from A, $F(A) = a_1$. Then from the set $A - \{a_1\}$, the choice function F chooses a_2, $F(A - \{a_1\}) = a_2$. The remaining set $A - \{a_1, a_2, \dots, a_n\}$ is non-empty because the set A is infinite.

● PROBLEM 5-13

Prove the theorems:

1. A subset of a denumerable set is countable.

2. Every subset of a countable set is countable.

SOLUTION:

1. Let A denote any denumerable set

$$A = \{a_1, a_2, a_3, \ldots\} \tag{1}$$

and let B denote a subset of A, $B \subset A$. When $B = \phi$, then of course B is finite. If $B \neq \phi$, then we can denote the least positive integer as k, such that

$$a_{k_1} \in B. \tag{2}$$

By k_2 we denote the least positive integer, such that

$$k_2 > k_1 \quad \text{and} \quad a_{k_2} \in B. \tag{3}$$

By repeating the procedure, we obtain

$$B = \{a_{k_1}, a_{k_2}, a_{k_3}, \ldots\}. \tag{4}$$

If the set of integers $\{k_1, k_2, \ldots\}$ is finite, then B is finite, otherwise B is denumerable.

2. Suppose A is a countable set, then A is either finite or denumerable. In either case, its subsets are countable.

● PROBLEM 5-14

Let A and B denote disjoint sets, $A \cap B = \phi$, and let A denote an infinite set and B denote a denumerable set. Show that

$$A \cup B \sim A. \tag{1}$$

SOLUTION:

Since A is an infinite set, it must contain a denumerable subset $P = \{p_1, p_2, p_3, \ldots\}$ (See Problem 5-12).

Let us write

$$A \cup B = (A - P) \cup (P \cup B) \quad \text{and} \quad A = (A - P) \cup P. \tag{2}$$

We shall establish an equivalence relation between $A \cup B$ and A in the following manner:

$$f(x) = \begin{cases} x & \text{for } x \in A - P \\ p_{2n-1} & \text{for } x = p_n \\ p_{2n} & \text{for } x = b_n \end{cases} \tag{3}$$

where

$$B = \{b_1, b_2, b_3, \ldots\} \tag{4}$$

Figure 1 illustrates function $f(x)$.

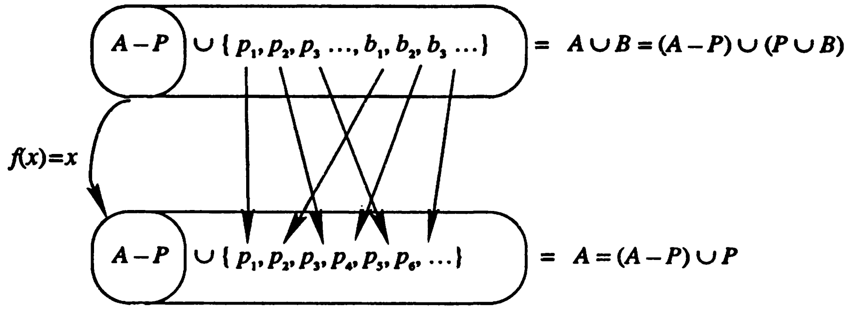


FIGURE 1

Function $f: A \cup B \to A$ is one-to-one and onto. Hence, the sets $A \cup B$ and A are equivalent

$$A \cup B \sim A. \qquad (5)$$

● PROBLEM 5-15

Prove the following important theorem:

THEOREM

Let $\{A_1, A_2, A_3, \ldots\}$ denote a countable sequence of countable sets, then the union

$$A = A_1 \cup A_2 \cup A_3 \cup \ldots \qquad (1)$$

is a countable set.

SOLUTION:

We proved (see Problem 5-9) that a Cartesian product of two countable sets is a countable set. From that, it follows that every double sequence can be transformed into a simple sequence. The elements of the array

$$\begin{matrix} a_{11}, a_{12}, \ldots, a_{1n}, \ldots \\ a_{21}, a_{22}, \ldots, a_{2n}, \ldots \\ \vdots \\ a_{m1}, a_{m2}, \ldots, a_{mn}, \ldots \end{matrix} \qquad (2)$$

can be written in the form of the infinite sequence. Each of the sets A_m is a countable set. Therefore,

$$A_m = \{a_{m1}, a_{m2}, a_{m3}, \dots\}. \tag{3}$$

We obtain a double sequence as in (2), because all A_m are countable. A double sequence (a_{mn}) can be transformed into a simple sequence with possible repetitions

$$a_{11}, a_{12}, a_{21}, a_{31}, \dots \tag{4}$$

We apply the axiom of choice, because the set of sequences consisting of the elements of the set A_m contains more than one element. We showed here that the elements of the set

$$A = A_1 \cup A_2 \cup A_3 \cup \dots = \bigcup_{n=1}^{\infty} A_n \tag{5}$$

can be arranged in a sequence. Therefore, A is a countable set.

● PROBLEM 5–16

> Prove that the set of all intervals on the real axis, with both endpoints rational numbers, is countable.
>
> Is the set of all oriented intervals on the real axis, with both endpoints rational numbers, countable as well?

SOLUTION:

The set of rational numbers Q is countable. Hence the Cartesian product $Q \times Q$ is a countable set. Let P denote the set of intervals

$$P = \{[a, b] : a \in Q \wedge b \in Q\}$$

with endpoints rational numbers. We define mapping f, such that

$$f : P \to Q \times Q$$

$$f : [a, b] \to (a, b) \in Q \times Q$$

When P is the set of oriented intervals, then f is one-to-one and onto. Indeed, for example

$$f([0,1]) = (0, 1)$$

$$f([1, 0]) = (1, 0).$$

Therefore, the set of oriented intervals is countable. From that, we conclude that the set of intervals is countable.

● PROBLEM 5–17

Show that the power set 2^A of A is equivalent to the collection of characteristic functions on A, $C(A)$, i.e. prove that

$$2^A \sim C(A). \tag{1}$$

SOLUTION:

Let B represent any subset of A,

$$B \in 2^A. \tag{2}$$

We define function

$$f: 2^A \to C(A) \tag{3}$$

in such a way that

$$f(B) = X_B = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}. \tag{4}$$

Then function f is one-to-one and onto. Therefore, the sets 2^A and $C(A)$ are equivalent.

$$2^A \sim C(A). \tag{5}$$

● PROBLEM 5–18

Prove that the set of spheres in a three-dimensional space, which have the rational coordinates of the center and rational radii, is countable.

SOLUTION:

The set Q of all rational numbers is countable.

Now, consider the set P of all points in a three-dimensional space, such that $p \in P$, if all three coordinates of p are rational numbers

$$P = \{p : p \in R^3; p = (p_1, p_2, p_3); p_1 \in Q \wedge p_2 \in Q \wedge p_3 \in Q\}.$$

It is easy to see that

$$P \sim Q \times Q \times Q.$$

Hence, the set of centers of the spheres is countable. The radii of the spheres are rational numbers.

Let Q^+ represent the set of all non-negative rational numbers. Then the set of all spheres, which have the rational coordinates of the center and

rational radii, is equivalent to the set $Q \times Q \times Q \times Q^+$. The Cartesian product of a finite number of countable sets is a countable set.

Therefore, $Q \times Q \times Q \times Q^+$ is a countable set and the set of spheres is countable.

● PROBLEM 5-19

Show that every set of disjoint intervals is countable.

SOLUTION:

Let P denote the set of disjoint intervals, and let $p \in P$ represent an interval

$$p = [\alpha, \beta] \tag{1}$$

where α and β are real numbers. We can always find an interval p', such that

$$p' = [a, b] \tag{2}$$

and $p' \subset p$, that is

$$[a, b] \subset [\alpha, \beta] \tag{3}$$

where a and b are rational numbers.

The set of rational numbers is countable. Hence, the set of all intervals with rational endpoints is countable.

Let $F : P \rightarrow P'$ represent a one-to-one and onto function, such that to every interval $[\alpha, \beta] \in P$, F assigns an interval $[a, b]$, such that $[a, b] \subset [\alpha, \beta]$ and a and b are rational numbers. Here P' is the set of disjoint intervals with rational endpoints. Since P' is a subset of a countable set (the set of all intervals with rational endpoints), P' is countable.

Therefore, P is also countable.

● PROBLEM 5-20

Prove that the set of proper maxima of the function is countable.

SOLUTION:

Let f represent a function with real arguments and values.

DEFINITION

Function f has a proper maximum at the point a, if an interval $[b, c]$ exists

such that $a \in \,]b, c[$, and such that for each $x \in \,]b, c[$ and $x \neq a$,

$$f(x) < f(a). \tag{1}$$

For each point x', which is the proper maximum of function $f(x)$, an interval $[b, c]$ with the properties described in the definition exists.

It is possible that to the proper maximum x' there corresponds an interval $[b', c']$ which contains another proper maximum (or maxima) as shown in Figure 1.

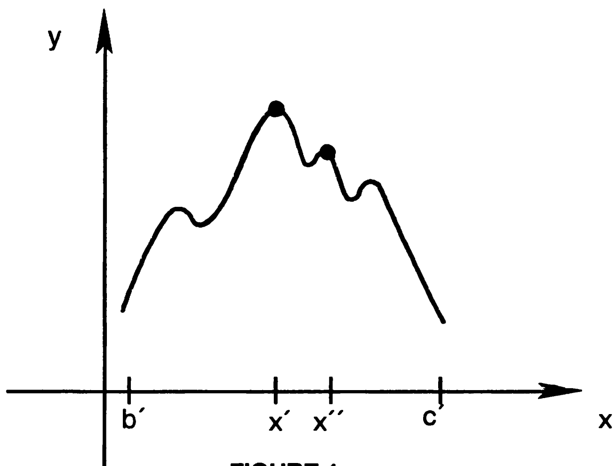


FIGURE 1

For each proper maximum, we can find an interval $[\alpha, \beta]$ with the rational endpoints which does not contain other proper maxima. Hence, we obtain a family of disjoint intervals, which is countable (see Problem 5-19).

Therefore, the set of proper maxima of the function is countable.

● PROBLEM 5–21

Prove that the set of points of discontinuity of a monotonic function in an interval $[a, b]$ is countable.

SOLUTION:

Suppose $f(x)$ is a monotonically increasing function. Function $f(x)$ is discontinuous at α, if and only if

$$p(\alpha) = f(\alpha + 0) - f(\alpha - 0) > 0 \tag{1}$$

where $f(\alpha + 0)$ denotes the right-hand limit, and $f(\alpha - 0)$ the left-hand limit. Of course, we should set

$$f(a - 0) = f(a) \quad \text{and} \quad f(b + 0) = f(b). \tag{2}$$

If

$$a < \alpha_1 < \alpha_2 < \ldots < \alpha_s < b \tag{3}$$

and

$$\alpha_\mu < x_\mu < \alpha_{\mu+1} \tag{4}$$

then

$$f(x_\mu) - f(x_{\mu-1}) \geq f(\alpha_\mu + 0) - f(\alpha_\mu - 0) = p(\alpha_\mu). \tag{5}$$

Hence

$$f(b) - f(a) = \sum_{\mu=1}^{s} [f(x_\mu) - f(x_{\mu-1})] \geq \sum_{\mu=1}^{s} p(\alpha_\mu) \tag{6}$$

where

$$x_0 = a \quad \text{and} \quad x_s = b. \tag{7}$$

When α_μ are numbers with $p(\alpha_\mu) > {}^1/_n$, it follows that

$$M < n\,[f(b) - f(a)]. \tag{8}$$

Therefore, the number of points of discontinuity α with $p(\alpha) > {}^1/_n$ has a fixed upper bound. Thus, a finite number of points of discontinuity with $p(\alpha) > {}^1/_n$ belong to the interval $[a, b]$.

Now we can write the finite number of points of discontinuity with $p(\alpha) > 1$, then $p(\alpha) > {}^1/_2$, then $p(\alpha) > {}^1/_3$, etc.

For each point of discontinuity α, n exists, such that $p(\alpha) > {}^1/_n$. Therefore, each point of discontinuity appears in the sequence. The set of points of discontinuity is countable.

● PROBLEM 5-22

Prove this rather amazing result: that the set of points of a straight line and the set of points of a space are equivalent, that is, that R^1 and R^3 are equivalent.

SOLUTION:

We shall first prove that the set of points of a cube of unit length and the set of points of an interval [0, 1] are equivalent. The cube is shown in Figure 1.

The coordinates of a point P interior to the cube are x, y z. Each of these real numbers can be written as a decimal fraction

$$x = 0.a_1a_2a_3 \ldots$$

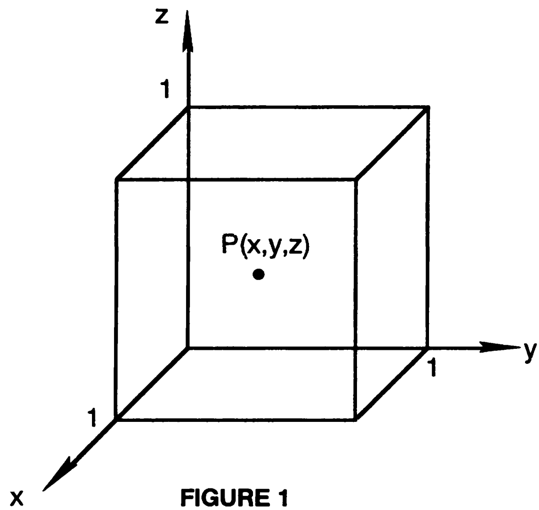


FIGURE 1

$$y = 0.b_1b_2b_3 \ldots$$

$$z = 0.c_1c_2c_3 \ldots$$

Let us form a real number p in the following manner:

$$p = 0.a_1b_1c_1a_2b_2c_2a_3b_3c_3 \ldots$$

Hence, to every point P interior to the cube a unique point p can be assigned, where $0 < p < 1$.

This is a one-to-one and onto mapping. Therefore, the sets of points of an interval and of a cube are equivalent. Since the interval and the whole real axis are equivalent, the result follows.

It is surprising that the sets of different dimensions are equivalent. A one-to-one and onto correspondence between them exists.

Hilbert, Peano and Brouwer proved that this correspondence cannot be continuous.

THEOREM (HILBERT, PEANO, BROUWER)

No one-to-one correspondence that maintains continuity exists between two continuums of different order.

Correspondence is continuous when the neighboring points of one continuum can be mapped to the neighboring points of the other continuum.

● PROBLEM 5-23

Show that, if $A \sim B$, then

$$A^C \sim B^C. \tag{1}$$

SOLUTION:

Sets A and B are equivalent, hence a function

$$g : A \to B \tag{2}$$

exists, which is a bijection (one-to-one and onto).

Let $f \in A^C$. We must define a function

$$G : A^C \to B^C \tag{3}$$

which is a bijection, to show that the sets A^C and B^C are equivalent.

Let us set (see Figure 1)

$$G(f) = g \circ f. \tag{4}$$

Function G is one-to-one, since

$$g \circ f = g \circ h \text{ implies } f = h.$$

Function G is also onto, because for $h \in B^C$, we obtain

$$G(g^{-1} \circ h) = g \circ g^{-1} \circ h = h \tag{5}$$

Since G is one-to-one and onto, it is bijective. Hence $A^C \sim B^C$.

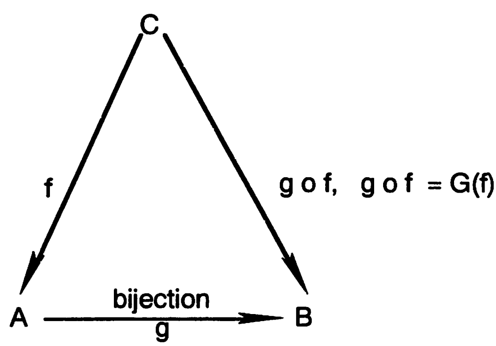


FIGURE 1

● PROBLEM 5–24

Show that

$$A^C \times B^C \sim (A \times B)^C. \tag{1}$$

SOLUTION:

We repeat

$$A^C = \{f : f : C \to A\}. \tag{2}$$

We define

$$G : A^C \times B^C \to (A \times B)^C \tag{3}$$

in such a way that

$$G(f, g)(x) = (f(x), g(x)) \text{ for all } x \in C. \tag{4}$$

Function G is one-to-one. Indeed let

$$G(f_1, g_1) = G(f_2, g_2). \tag{5}$$

Then, for all $x \in C$

$$G(f_1, g_1)(x) = G(f_2, g_2)(x) \tag{6}$$

and

$$(f_1(x), g_1(x)) = (f_2(x), g_2(x)). \tag{7}$$

Hence

$$f_1(x) = f_2(x) \text{ for all } x \in C$$

and

$$g_1(x) = g_2(x) \text{ for all } x \in C.$$

Thus

$$f_1 = f_2 \quad \text{and} \quad g_1 = g_2,$$

$$(f_1, g_1) = (f_2, g_2). \tag{8}$$

Now we shall prove that G is onto. Let $h \in (A \times B)^C$,

$$h(x) \in A \times B \text{ for all } x \in C.$$

Let

$$(h_1(x), h_2(x)) = h(x). \tag{9}$$

Then

$$h_1 \in A^C \quad \text{and} \quad h_2 \in B^C$$

$$(h_1, h_2) \in A^C \times B^C. \tag{10}$$

Thus

$$G(h_1, h_2) = h \quad \text{since} \tag{11}$$

$$G(h_1, h_2)(x) = (h_1(x), h_2(x)) = h(x) \tag{12}$$

for all $x \in C$.

Function $G : A^C \times B^C \to (A \times B)^C$ is bijective, hence, the sets $A^C \times B^C$ and $(A \times B)^C$ are equivalent.

CHAPTER 6

CARDINAL NUMBERS, CARDINAL ARITHMETIC

● PROBLEM 6–1

The power of the set of natural numbers is denoted by card N (or $\aleph$) and the power of the set of real numbers is denoted by card R or by C (continuum). These are the most important of the infinite cardinal numbers. The following important inequality holds:

$$\text{card } R \neq \text{card } N. \tag{1}$$

Prove (1) by applying the results of Chapter 5.

SOLUTION:

In Chapter 5, we showed that for every sequence of real numbers a_1, a_2, a_3, ..., a_n, ..., we can define a real number b, which does not belong to this sequence. This can be formulated in the form of a theorem.

THEOREM

The set of all real numbers is non-countable.

Therefore, a cardinal number assigned to the set of natural numbers, card N, is different from the cardinal number assigned to the set of real numbers, card R.

$$\text{card } R \neq \text{card } N$$

The cardinal number assigned to the set A is denoted by card A.

● PROBLEM 6–2

In Problem 3-26, we discussed the axiomatic formulation of set theory. Describe how the concept of cardinal numbers can be established axiomatically.

SOLUTION:

The cardinal numbers describe the "size" of the sets. In this sense, they can be considered a generalization of natural numbers. To the set of, say, five books, we assign the natural number five, which describes how large the set is.

The axiomatic formulation given here does not define a cardinal number and is not related to counting. The concept of a cardinal number is related to "size." We want to determine if one of two given sets has more members than the other. Thus, we do not have to count; just pair off each member of

one set with a member of the other and see if any elements are left over.

The following axioms introduce the concept of cardinal numbers:

I. Each set X is associated with a cardinal number, denoted by card X, and for each cardinal number p, a set X exists such that

$$\text{card } X = p.$$

II. Card $X = 0$, if and only if $X = \phi$.

III. If X is a non-empty finite set, that is, $X \sim \{1, 2, \ldots, n\}$ for some $n \in N$, then card $X = n$.

IV. For any two sets X and Y, card X = card Y, if and only if $X \sim Y$.

Axioms I and IV are usually called the axiom of cardinality.

From II and III, we see that the cardinal number of a finite set is the number of elements of that set.

● PROBLEM 6-3

Explain why

$$\text{card } N < \text{card } R \tag{1}$$

where N is the set of positive integers and R is the set of real numbers.

SOLUTION:

The cardinal number of a finite set is called a finite cardinal number. Finite cardinal numbers are actually non-negative integers. The cardinal number of an infinite set is called a transfinite cardinal number. Finite cardinal numbers have the inherited order of natural numbers

$$0 < 1 < 2 < \ldots < n < n + 1 < \ldots \tag{2}$$

The question is how to compare two transfinite cardinal numbers. By applying axiom IV (see Problem 6-2), we can determine whether two cardinal numbers are equal or not. To establish which of the two cardinal numbers is "larger," we shall use the following:

DEFINITION

Let A and B represent two sets and card A and card B denote their cardinal numbers, respectively. Then card A is said to be less than card B; we write

$$\text{card}\, A < \text{card}\, B \tag{3}$$

when the set A is equipotent to a subset of B, but the set B is not equipotent to any subset of A.

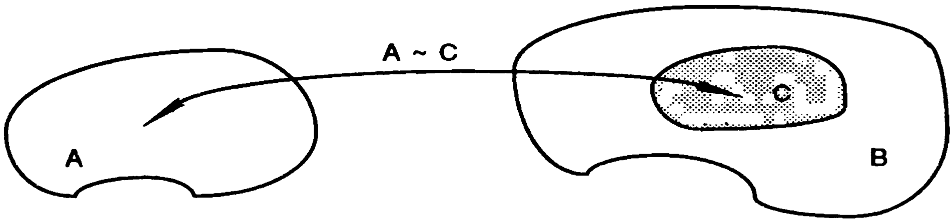


FIGURE 1

D does not exist, such that $D \subset A$ and $D \sim B$.

The set N is a subset of R, hence N is equipotent to a subset of R

$$N \sim N \subset R. \tag{4}$$

The infinite set R is nondenumerable. Thus, R is not equipotent to any subset of N. Therefore,

$$\text{card}\, N < \text{card}\, R. \tag{5}$$

● PROBLEM 6-4

Let m denote any finite cardinal number. Prove that

$$m < \text{card}\, N. \tag{1}$$

SOLUTION:

Let A represent the subset of the set of natural numbers N.

$$A = \{1, 2, \ldots, m\} \tag{2}$$

Then

$$\text{card}\, A = m \quad \text{and} \quad A \subset N. \tag{3}$$

We have

$$m = \text{card}\, A < \text{card}\, N \tag{4}$$

because the set A is equipotent to a subset of N, but N is not equipotent to any subset of A (see Problem 6-3).

Prove the following:

LEMMA

If B is a subset of A, and if an injection (one-to-one) $f : A \rightarrow B$ exists, then there is a bijection (one-to-one and onto) $g : A \rightarrow B$.

SOLUTION:

If $B = A$, then the identity function is one-to-one and onto (bijection). Suppose B is a proper subset of A. Let us denote the set

$$C = \bigcup_{n=0}^{\infty} f^n(A - B) \tag{1}$$

by C where f^0 is the identify function on A and

$$f^n(x) = f(f^{n-1}(x)) \tag{2}$$

for each $x \in A$ and for each positive integer n. Note that $A - B \subset C$ and $f(C) \subset C$. For any two distinct non-negative integers m,n, the sets $f^m(A - B)$ and $f^n(A - B)$ are disjoint.

Indeed, suppose $f^m(A - B) \cap f^n(A - B) \neq \phi$, then $x_1, x_2 \in A - B$ exists, such that

$$f^m(x_1) = f^n(x_2) \tag{3}$$

and

$$f^m(x_1) = f^{n-m}(f^m(x_2)) = f^m(f^{n-m}(x_2)). \tag{4}$$

Function f is one-to-one, hence

$$f^{n-m}(x_2) = x_1 \in (A - B) \cap B \tag{5}$$

is a contradiction.

We define function $g : A \rightarrow B$

$$g(x) = \begin{cases} f(x) & \text{for } x \in C \\ x & \text{for } x \in A - C \end{cases} \tag{6}$$

Function g is one-to-one, also:

$$\begin{aligned} g(A) &= f(C) \cup (A - C) = \\ &= f\left(\bigcup_{n=0}^{\infty} f^n(A - B)\right) \cup \left(A - \bigcup_{n=0}^{\infty} f^n(A - B)\right) = \\ &= \left(\bigcup_{n=1}^{\infty} f^n(A - B)\right) \cup \left(A - \bigcup_{n=0}^{\infty} f^n(A - B)\right) = \\ &= A - (A - B) = B. \end{aligned} \tag{7}$$

Remember f^0 is the identity function. Thus, function g is a bijection. The diagram illustrates the proof.

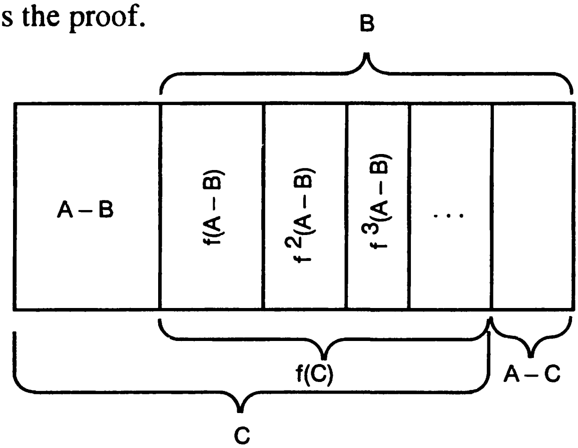


FIGURE 1

The set A is represented by the whole rectangle.

● PROBLEM 6-6

So far, we have considered the following cases:

1. If two sets A and B are equipotent, then

$$\text{card}\, A = \text{card}\, B. \tag{1}$$

2. Set A is equipotent to a subset of B, but the set B is not equipotent to any subset of A, then

$$\text{card}\, A < \text{card}\, B. \tag{2}$$

The question is how two cardinal numbers, card A and card B, compare when A is equipotent to a subset of B and B is equipotent to a subset of A. Georg Cantor conjectured that in such a case

$$\text{card}\, A = \text{card}\, B. \tag{3}$$

It was later independently proven by both E. Schröder and F. Bernstein. Prove:

SCHRÖDER-BERNSTEIN THEOREM

If A and B are sets, such that A is equipotent to a subset of B and B is equipotent to a subset of A, then A and B are equipotent, i.e.

$$\text{card}\, A = \text{card}\, B.$$

SOLUTION:

Suppose A_1` $\subset A$ and $B_1 \subset B$ and

$$A_1 \sim B, \; B_1 \sim A. \tag{4}$$

Then two bijections exist

$$f_1 : A \to B_1 \quad \text{and} \quad g_1 : B \to A_1. \tag{5}$$

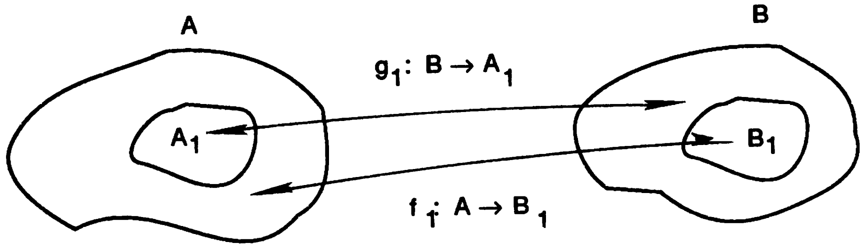


FIGURE 1

Let us define

$$f : A \to A_1 \text{ by } f(x) = g_1(f_1(x)). \tag{6}$$

Then f is one-to-one. By the lemma proved in Problem 6-5, a bijection $g : A \to A_1$ exists.

Therefore, since $g : A \to A_1$ is a bijection and $g_1^{-1} : A_1 \to B$ is a bijection, their composition

$$g_1^{-1} \circ g : A \to B \tag{7}$$

is a bijection. Hence, $A \sim B$ and

$$\text{card } A = \text{card } B. \tag{8}$$

● PROBLEM 6-7

Prove the following version of the Schröder-Bernstein theorem:

THEOREM

If card $A \le$ card B and card $B \le$ card A, then card A = card B.

SOLUTION:

We shall write card $A \le$ card B to mean card $A <$ card B or card A = card B.

Suppose two sets are given, A and B. By applying the Schröder-Bernstein theorem, we conclude that one of three possibilities takes place:

$$\operatorname{card} A < \operatorname{card} B$$

or

$$\operatorname{card} A > \operatorname{card} B$$

or

$$\operatorname{card} A = \operatorname{card} B.$$

Hence, an immediate consequence of the Schröder-Bernstein theorem is

$$\begin{pmatrix} \operatorname{card} A \leq \operatorname{card} B \\ \operatorname{card} B \leq \operatorname{card} A \end{pmatrix} \Rightarrow \left(\operatorname{card} A = \operatorname{card} B \right)$$

● PROBLEM 6–8

Prove that $\operatorname{card} N$ is the smallest transfinite cardinal number.

SOLUTION:

The cardinal number assigned to a finite set is called a finite cardinal number and the cardinal number assigned to an infinite set is called a transfinite cardinal number. We already know two transfinite cardinal numbers, $\operatorname{card} N$ and $\operatorname{card} R$, where

$$\operatorname{card} N < \operatorname{card} R. \tag{1}$$

Thus

$$\underbrace{0 < 1 < 2 < \ldots}_{\text{Finite Cardinal Numbers}} \qquad \underbrace{< \ldots ? \ldots < \operatorname{card} N < \ldots ? \ldots < \operatorname{card} R \ \ldots ? \ldots}_{\text{Transfinite Cardinal Numbers}}$$

FIGURE 1

Figure 1 illustrates the state of our knowledge about cardinal numbers. Each question mark indicates that we are not sure if the respective cardinal number exists.

Let A represent an infinite set. We shall apply the following:

THEOREM

Every infinite set contains a denumerable subset.

Therefore, a set B exists, such that

$$B \subset A \text{ and } B \sim N. \tag{2}$$

Thus

$$\text{card } B = \text{card } N \tag{3}$$

and

$$\text{card } B \leq \text{card } A. \tag{4}$$

We conclude, that for every infinite set A,

$$\text{card } N \leq \text{card } A. \tag{5}$$

The number, card N, is the smallest transfinite cardinal number. We can eliminate one of the question marks; see Figure 2.

$$\underbrace{0 < 1 < 2 < \ldots}_{\text{Finite Cardinal Numbers}} \quad \underbrace{< \text{card } N < \ldots ? \ldots < \text{card } R < \ldots ? \ldots}_{\text{Transfinite Cardinal Numbers}}$$

FIGURE 2

We shall eliminate the remaining question marks later.

● PROBLEM 6–9

Prove the following:

$$\begin{pmatrix} A \subset B \subset C \\ A \sim C \end{pmatrix} \Rightarrow (A \sim B).$$

SOLUTION:

We shall apply the Schröder-Bernstein theorem. Since $A \subset B \subset C$, we have

$$B \sim B \subset C \tag{1}$$

and

$$A \subset B. \tag{2}$$

But $A \sim C$, therefore

$$B \sim B \subset C \tag{3}$$

$$C \sim A \subset B \tag{4}$$

as shown in Figure 1.

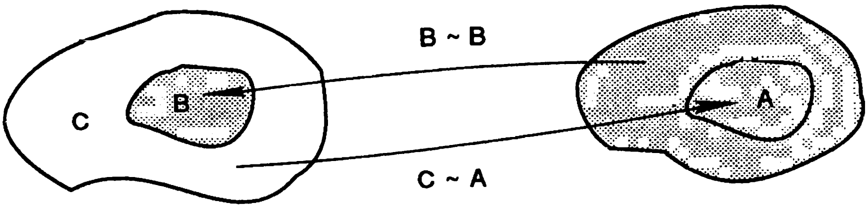


FIGURE 1

By the Schröder-Bernstein theorem we conclude that $C \sim B$. Therefore, since $A \sim C$ and $C \sim B$, we obtain $A \sim B$.

● PROBLEM 6–10

Show that the largest cardinal number does not exist.

SOLUTION:

Georg Cantor proved the following theorem:

CANTOR'S THEOREM

For any set X,

$$\text{card } X < \text{card } P(X). \tag{1}$$

Remember that the power set $P(X)$ of X is the set of all subsets of X.

Let X represent a set and $P(X)$ its power set. Then, by applying Cantor's theorem

$$\text{card } X < \text{card } P(X). \tag{2}$$

Since $P(X)$ is again the set, we have

$$\text{card } P(X) < \text{card } P(P(X)). \tag{3}$$

By applying Cantor's theorem, we obtain

$$\text{card } X < \text{card } P(X) < \text{card } P(P(X)) < \dots \tag{4}$$

In such a way, we can obtain a sequence of cardinal numbers.

By setting $X = R$, we obtain a sequence of transfinite cardinal numbers

$$\text{card } R < \text{card } P(R) < \text{card } P(P(R)) < \dots \tag{5}$$

Thus, there is no largest cardinal number.

● PROBLEM 6–11

Let A and B represent sets. Prove that if $A \sim B$, then

$$\text{card } P(A) = \text{card } P(B). \tag{1}$$

SOLUTION:

Since $A \sim B$, a bijection exists

$$f: A \to B. \tag{2}$$

We shall show that a bijection exists

$$F: P(A) \to P(B). \tag{3}$$

Let us define

$$F(X) = f(X) \text{ for all } X \in P(A). \tag{4}$$

Since f is a bijective function, F is also. Hence,

$$P(A) \sim P(B). \tag{5}$$

● PROBLEM 6–12

Let X represent a denumerable set. Prove that the power set $P(X)$ of X is nondenumerable.

SOLUTION:

We shall apply Cantor's theorem. Suppose, on the contrary, that a denumerable set X exists, such that its power set $P(X)$ is also denumerable. Then

$$\text{card } X = \text{card } P(X) \tag{1}$$

but this contradicts Cantor's theorem.

That completes the proof.

● PROBLEM 6–13

Prove the following:

CANTOR'S THEOREM

If A is a set, then

$$\operatorname{card} A < \operatorname{card} P(A). \tag{1}$$

SOLUTION:

If $A = \phi$, then card $\phi = 0$ and card $P(\phi) = 1$.

Suppose $A \neq \phi$. Let us define a function

$$f : A \to P(A) \tag{2}$$

such that

$$f(x) = \{x\} \in P(A) \tag{3}$$

for all $x \in A$.

Function f is one-to-one. Therefore, the sets A and $\{\{x\} : x \in A\}$ are equipotent. But

$$\{\{x\} : x \in A\} \subset P(A). \tag{4}$$

Therefore,

$$\operatorname{card} A \leq \operatorname{card} P(A). \tag{5}$$

Now we must show that the sets A and $P(A)$ are not equipotent, i.e. that card $A \neq$ card $P(A)$. Assume, on the contrary, that the sets A and $P(A)$ are equipotent, hence a bijection exists

$$g : A \to P(A). \tag{6}$$

Let us define the set

$$T = \{x \in A : x \notin g(x)\}. \tag{7}$$

Since $T \in P(A)$ and $g : A \to P(A)$, an element $a \in A$ exists, such that

$$g(a) = T. \tag{8}$$

The existence of such an element leads to a contradiction. Because if $a \in T$, then $a \notin g(a)$, according to the definition of T; on the other hand, $g(a) = T$ and $a \in T$.

If $a \notin T$, then since $g(a) = T$, we have $a \notin g(a)$ and $a \in T$ and consequently, $a \in g(a)$.

That completes the proof of Cantor's theorem.

● PROBLEM 6–14

Let α and β denote two cardinal numbers. Define their sum and show that the sum is uniquely defined, i.e. is independent of the choice of sets.

SOLUTION:

The sum $\alpha + \beta$ of two cardinal numbers α and β is defined to be the power of the union of two disjoint sets, which have the powers α and β, respectively.

We have

$$\operatorname{card} A + \operatorname{card} B = \operatorname{card}(A \cup B), \text{ if } A \cap B = \phi. \tag{1}$$

The cardinal number assigned to set X we denote by $\operatorname{card} X$.

Sometimes the other notations are used, like $\overline{\overline{X}}$, $\#X$, etc.

In this definition, we employ two disjoint sets. Suppose A_1 and B_1 have powers, $\operatorname{card} A_1$ and $\operatorname{card} B_1$, respectively. Then, the sets A and B exist, which are disjoint and such that

$$\operatorname{card} A_1 = \operatorname{card} A \text{ and } \operatorname{card} B_1 = \operatorname{card} B. \tag{2}$$

Indeed, suppose $a \neq b$ are two distinct elements, then

$$A = \{a\} \times A_1 \quad B = \{b\} \times B_1$$

$$\text{and } \; A \cap B = \phi. \tag{3}$$

Since

$$\begin{pmatrix} A_1 \sim B_1 \\ A_2 \sim B_2 \\ A_1 \cap A_2 = B_1 \cap B_2 = \phi \end{pmatrix} \Rightarrow (A_1 \cup A_2 \sim B_1 \cup B_2),$$

we conclude that for every two cardinal numbers, their sum is uniquely defined, that is, independent of the choice of sets A and B.

Remember that $A \sim B$ indicates that sets A and B have the same power (are equipotent).

● PROBLEM 6-15

Find the sum of cardinal numbers

$$\operatorname{card} N + \operatorname{card} N. \tag{1}$$

SOLUTION:

Let N_O and N_E denote the set of odd natural numbers and the set of even natural numbers. We have

$$N_O \cap N_E = \phi \tag{2}$$

$$N_O \subset N, \quad N_E \subset N \tag{3}$$

$$N_O \cup N_E = N. \tag{4}$$

The sets N_O and N_E are denumerable. Hence,

$$\text{card}\, N + \text{card}\, N = \text{card}\, N_O + \text{card}\, N_E =$$

$$= \text{card}\,(N_O \cup N_E) = \text{card}\, N. \tag{5}$$

Similarly, card R + card R = card R. It is easy to show that card N + card R = card R. Indeed

$$(0,1) \sim R \quad \text{and} \quad \text{card}\,(0,1) = \text{card}\, R. \tag{6}$$

Let $P = (0, 1) \cup N$. Since N and $(0,1)$ are disjoint sets,

$$\text{card}\, P = \text{card}\,(0,1) + \text{card}\, N = \text{card}\, N + \text{card}\, R\,. \tag{7}$$

On the other hand,

$$R \sim (0,1) \subset P \tag{8}$$

and

$$P \sim P \subset R. \tag{9}$$

According to the Schröder-Bernstein theorem,

$$P \sim R \quad \text{and} \quad \text{card}\, P = \text{card}\, N + \text{card}\, R = \text{card}\, R. \tag{10}$$

The power of the set of natural numbers will be denoted throughout this book by card N or by $\aleph_0$ (aleph-null).

● PROBLEM 6–16

Define the product of two cardinal numbers and show that the product is uniquely defined.

SOLUTION:

Let α and β denote two cardinal numbers and let A and B represent two sets, such that

$$\text{card}\, A = \alpha \quad \text{and} \quad \text{card}\, B = \beta. \tag{1}$$

Then, we define the product $\alpha \cdot \beta$ of α and β to be the power of the Cartesian product of sets A and B:

$$\text{card}\, A \cdot \text{card}\, B = \text{card}\,(A \times B). \tag{2}$$

This definition is unique, in the sense that it does not depend on the choice of sets A and B. Suppose A_1 and B_1 are two sets, such that

$$A \sim A_1 \quad \text{and} \quad B \sim B_1 \tag{3}$$

Since the following holds

$$\begin{pmatrix} A \sim A_1 \\ B \sim B_1 \end{pmatrix} \Rightarrow (A \times B \sim A_1 \times B_1) \tag{4}$$

we conclude that the definition of the product of two cardinal numbers is unique.

● PROBLEM 6–17

Show that

$$\text{card}\, N + \text{card}\, N = \text{card}\, N \tag{1}$$

$$\text{card}\, N \cdot \text{card}\, N = \text{card}\, N \tag{2}$$

$$\text{card}\, N + n = \text{card}\, N \tag{3}$$

$$\text{card}\, N \cdot n = \text{card}\, N \tag{4}$$

where n is a natural number and card N is the power of the set of natural numbers.

SOLUTION:

In Chapter 5, we proved the following useful theorem:

THEOREM

The union $A \cup B$ of two countable sets A and B is countable.

Therefore, since the set of natural numbers is countable, we conclude that

$$\text{card}\, N + \text{card}\, N = \text{card}\, N \tag{5}$$

From this theorem, we conclude as well that

$$\text{card}\, N + n = \text{card}\, N \tag{6}$$

To prove (2) we shall apply:

THEOREM

The Cartesian product of two countable sets is a countable set.

Therefore,

$$\text{card } N \cdot \text{card } N = \text{card } N \tag{7}$$

and similarly,

$$\text{card } N \cdot n = \text{card } N. \tag{8}$$

● PROBLEM 6–18

Show that addition and multiplication satisfy the associative, commutative and distributive laws.

SOLUTION:

Since

$$A \cup (B \cup C) = (A \cup B) \cup C \quad \text{and} \quad A \cup B = B \cup A$$

addition is associative and commutative.

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$

$$\alpha + \beta = \beta + \alpha \tag{1}$$

Similarly, since

$$A \times (B \times C) = (A \times B) \times C \tag{2}$$

multiplication is associative.

By applying the following formula

$$A \times B \sim B \times A \tag{3}$$

we conclude that multiplication is commutative

$$\alpha \cdot \beta = \beta \cdot \alpha. \tag{4}$$

To prove the distributive law

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma \tag{5}$$

let us choose three sets A, B and C, such that

$$\alpha = \text{card } A, \ \beta = \text{card } B, \ \gamma = \text{card } C \tag{6}$$

where $B \cap C = \phi$. Then

$$A \times (B \cup C) = A \times B \cup A \times C \tag{7}$$

$$(A \times B) \cap (A \times C) = A \times (B \cap C) = \phi \tag{8}$$

and

$$\text{card}\,[A \times (B \cup C)] = \text{card}\,A \times B + \text{card}\,A \times C \tag{9}$$

which proves (5).

● PROBLEM 6–19

Let x denote an arbitrary cardinal number. Compute the following cardinal numbers:

1. $0x$

2. $1x$.

SOLUTION:

1. Let X represent a set, such that

$$\text{card}\,X = x. \tag{1}$$

Then

$$X \times \phi = \phi. \tag{2}$$

Hence

$$0x = \text{card}\,\phi \cdot \text{card}\,X =$$

$$= \text{card}\,(\phi \times X) = \text{card}\,\phi = 0. \tag{3}$$

2. Let $\{a\}$ represent any set consisting of one element. Then the sets X and $X \times \{a\}$ are equipotent

$$X \sim X \times \{a\}. \tag{4}$$

We obtain

$$1x = \text{card}\,\{a\} \cdot \text{card}\,X =$$

$$= \text{card}\,\{a\} \times X = \text{card}\,X = x. \tag{5}$$

● PROBLEM 6–20

Which of the following theorems is true?

1. If x, y, z are cardinal numbers, such that $x \leq y$, then $xz \leq yz$.

2. If x, y and z are cardinal numbers, such that $x < y$, $z \neq 0$, then $xz < yz$.

SOLUTION:

1. Let A, B, and C represent the sets, such that

$$\text{card } A = x, \text{ card } B = y, \text{ card } C = z. \tag{1}$$

Since

$$\text{card } A \leq \text{card } B \tag{2}$$

a one-to-one function exists, such that

$$f : A \to B. \tag{3}$$

Let $g : C \to C$, $g(c) = c$ for every $c \in C$. Then function

$$F : A \times C \to B \times C \tag{4}$$

defined by

$$F(a, c) = (f(a), g(c)) \tag{5}$$

is also one-to-one. Therefore,

$$\text{card } (A \times C) \leq \text{card } (B \times C) \tag{6}$$

or

$$\text{card } A \cdot \text{card } C = xz \leq \text{card } B \cdot \text{card } C = yz. \tag{7}$$

2. This theorem is not true. The following example proves why:
Let

$$x = 1 \quad y = z = \text{card } N \tag{8}$$

then $x < y$ but

$$xz = 1 \cdot \text{card } N = \text{card } N \quad \text{and} \quad yz = \text{card } N \, \text{card } N = \text{card } N.$$

Hence $xz = yz$.

● PROBLEM 6–21

Prove that

$$\text{card } R \cdot \text{card } R = \text{card } R. \tag{1}$$

SOLUTION:

In Chapter 5 we proved that the set of real numbers R and the open interval (0, 1) are equipotent

$$R \sim (0, 1). \tag{2}$$

To evaluate card R · card R, let us note that

$$\text{card } R = \text{card } (0, 1) \tag{3}$$

$$\text{card } R \cdot \text{card } R =$$

$$= \text{card } (0,1) \cdot \text{card } (0,1) = \text{card } (0,1) \times (0,1). \tag{4}$$

Hence, we have to compute card $(0,1) \times (0,1)$. To do it, a function f will be defined, such that

$$f : (0,1) \times (0,1) \to (0,1) \tag{5}$$

and f is bijective (i.e. one-to-one and onto).

Each $x \in (0,1)$ can be expressed by its infinite decimal expansion. For example

$$^2/_7 = 0.28571428\ldots$$

$$^1/_2 = 0.4999999\ldots \tag{6}$$

We define f in the following manner:

$$x, y \in (0,1)$$

$$f(x, y) = f(.x_1x_2x_3\ldots, .y_1y_2y_3\ldots) =$$

$$= .x_1y_1x_2y_2x_3y_3\ldots \tag{7}$$

It is easy to verify that such a function is one-to-one and onto. Thus

$$(0,1) \times (0,1) \sim (0,1) \tag{8}$$

and

$$\text{card } (0,1) \times (0,1) = \text{card } (0,1). \tag{9}$$

From (4) and (9), we obtain

$$\text{card } R \cdot \text{card } R = \text{card } (0,1) = \text{card } R. \tag{10}$$

• PROBLEM 6-22

Let x and y denote cardinal numbers. Prove that

1. If xy = 0 then $x = 0$ or $y = 0$.

2. If $xy = 1$ then $x = 1$ or $y = 1$.

SOLUTION:

1. Let A and B represent two sets, such that

$$\operatorname{card} A = x$$

$$\operatorname{card} B = y \tag{1}$$

then

$$xy = \operatorname{card}(A \times B) = 0. \tag{2}$$

Hence $A \times B$ is an empty set

$$A \times B = \phi \tag{3}$$

From (3), we conclude that

$$A = \phi \quad \text{and} \quad B = \phi. \tag{4}$$

Thus

$$\operatorname{card} A = x = 0$$

$$\operatorname{card} B = y = 0. \tag{5}$$

2.
$$xy = \operatorname{card} A \cdot \operatorname{card} B =$$

$$= \operatorname{card}(A \times B) = 1. \tag{6}$$

Set $A \times B$ consists of one element. Thus, each of the sets A and B consists of one element. We have

$$\operatorname{card} A = x = 1$$

$$\operatorname{card} B = y = 1. \tag{7}$$

● PROBLEM 6–23

Define b^a (a^{th} power of b), where a and b are cardinal numbers, and justify this definition using an analogy with natural numbers.

SOLUTION:

Let us start with a definition:

DEFINITION

Let $a \neq 0$ and b denote cardinal numbers and A and B represent two sets such that

$$\text{card}\, A = a, \quad \text{card}\, B = b \tag{1}$$

Then, we define b^a by

$$b^a = card\, B^A \tag{2}$$

where B^A denotes the set of all functions from A to B.

Consider the finite case of two natural finite numbers m and n. Then

$$n^m = n \cdot n \cdot n \cdot \ldots \cdot n, \ m \text{ times.} \tag{3}$$

Let A represent a set with m elements and B a set with n elements.

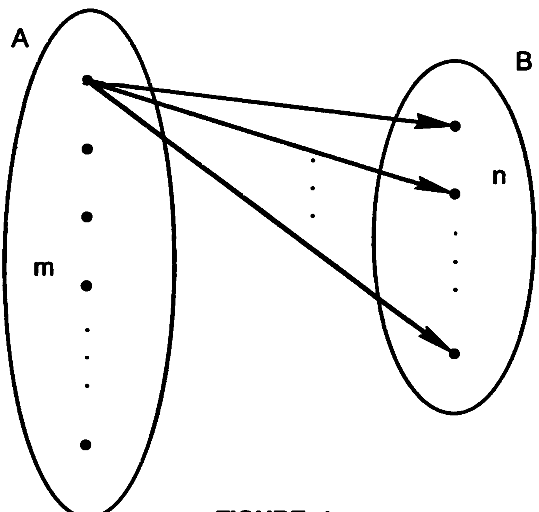


FIGURE 1

Each of the m elements of A has n choices for its image. The choices for each element of A are independently made and, since A consists of m elements, there are

$$n \cdot n \cdot n \cdot \ldots \cdot n = n^m$$

functions from A to B.

The definition of b^a is a generalization of this concept.

● PROBLEM 6–24

Show that the definition of b^a (see Problem 6-23) is independent of the choice of representatives, A and B.

SOLUTION:

We shall prove the following:

THEOREM

Let A, B, A', B' represent sets, such that $A \sim A'$ and $B \sim B'$. Then

$$B^A \sim B'^{A'} \tag{1}$$

Proof:

Since $A \sim A'$ and $B \sim B'$ are bijective functions, g and h exist

$$g : A \to A', \qquad h : B \to B'. \tag{2}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ A' \arrow[r, "F(f)"'] & B' \end{tikzcd}$$

FIGURE 1

Suppose $f : A \to B$ is a function, $f \in B^A$. Let us define

$$F : B^A \to B'^{A'} \tag{3}$$

by $F(f) : A' \to B'$

$$F(f)\,(x) = h(f(g^{-1}(x))) = h \circ f \circ g^{-1}(x). \tag{4}$$

Functions g and h are bijective, hence $F(f)$ is bijective. Therefore,

$$F : B^A \to B'^{A'} \tag{5}$$

is bijective and

$$B^A \sim B'^{A'} \tag{6}$$

● PROBLEM 6–25

Show that for any set A

$$\text{card } P(A) = 2^{\text{card } A}. \tag{1}$$

SOLUTION:

Let $B = \{0, 1\}$. To each subset X of A, $X \subset A$, we assign the characteristic function

$$X_x : A \to B \tag{2}$$

as follows:

$$X_x(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \in A - X \end{cases} \tag{3}$$

and a one-to-one and onto mapping exists between the sets $P(A)$ and B^A.

Remember that $P(A)$ is the set of all subsets of A, and B^A is the set of all functions from A to $\{0, 1\}$. To each subset $X \subset A$, $X \in P(A)$, we assign its characteristic function X_x, $X_x \in B^A$. Two different subsets cannot have the same characteristic function, hence mapping is one-to-one. Any characteristic function uniquely defines a subset, hence mapping is onto.

Bijection exists between $P(A)$ and B^A. Thus

$$\text{card } P(A) = \text{card } B^A \tag{4}$$

Since $B = \{0, 1\}$, card $B = 2$ and

$$\text{card } P(A) = 2^{\text{card } A}. \tag{5}$$

● PROBLEM 6–26

Let a, x, and y denote cardinal numbers. Show that

$$a^x a^y = a^{x+y}. \tag{1}$$

SOLUTION:

First of all, let us choose the sets A, X, Y, such that

$$\text{card } A = a$$

$$\text{card } X = x$$

$$\text{card } Y = y$$

$$X \cap Y = \phi. \tag{2}$$

From the definition of the sum of cardinal numbers, we have

$$\text{card } (X \cup Y) = x + y. \tag{3}$$

Now we shall show that the sets

$$A^X \times A^Y \text{ and } A^{X \cup Y}$$

are equipotent.

Suppose $f \in A^X$ and $g \in A^Y$, then $(f, g) \in A^X \times A^Y$. Then the function $f \cup g$ belongs to $A^{X \cup Y}$.

The union $f \cup g$ is the function

$$f \cup g : X \cup Y \to A \tag{4}$$

defined by

$$f \cup g(x) = \begin{cases} f(x) & \text{if} \quad x \in X \\ g(x) & \text{if} \quad x \in Y \end{cases} \tag{5}$$

Thus, we have established a one-to-one and onto relationship between the sets $A^X \times A^Y$ and $A^{X \cup Y}$.

Hence,

$$\begin{aligned} \operatorname{card} A^X \times A^Y &= a^x a^y = \\ &= \operatorname{card} A^{X \cup Y} = a^{x+y}. \end{aligned} \tag{6}$$

● PROBLEM 6–27

Let a, x, and y denote cardinal numbers. Show that

$$(a^x)^y = a^{xy}. \tag{1}$$

SOLUTION:

Let A, X, and Y represent sets, such that

$$\operatorname{card} A = a, \operatorname{card} X = x, \operatorname{card} Y = y. \tag{2}$$

We shall prove that the sets $A^{X \times Y}$ and $(A^X)^Y$ are equipotent. Let $f : X \times Y \to A$ represent a given function and b a given element of Y, $b \in Y$.

Then a function exists

$$f^b : X \to A \tag{3}$$

defined by

$$f^b(a) = f(a, b) \tag{4}$$

for all $a \in X$.

Now, let us define the function

$$F : A^{X \times Y} \to (A^X)^Y \tag{5}$$

such that to each $f \in A^{X \times Y}$, we assign the function

$$p_f \in (A^X)^Y \tag{6}$$

defined by

$$p_f(b) = f^b \tag{7}$$

for all $b \in Y$.

Function F is one-to-one and onto, hence the sets $A^{X \times Y}$ and $(A^X)^Y$ are equivalent. Thus

$$(a^x)^y = a^{xy}. \tag{8}$$

● PROBLEM 6–28

Prove the following:

THEOREM

If a, b, and x are cardinal numbers, then

$$(ab)^x = a^x b^x. \tag{1}$$

SOLUTION:

Let A, B, and X represent sets, such that

$$\text{card}\, A = a, \text{card}\, B = b, \text{card}\, X = x. \tag{2}$$

We must prove that the sets $(A \times B)^X$ and $A^X \times B^X$ are equipotent.

Recall that the A-projection

$$p_A : A \times B \to A \tag{3}$$

is a function which assigns element $a \in A$ to each ordered pair $(a, b) \in A \times B$. Let us define a bijection

$$F : (A \times B)^X \to A^X \times B^X. \tag{4}$$

Function F assigns to each

$$f \in (A \times B)^X, f : X \to A \times B \tag{5}$$

a function defined by

$$(p_A \circ f, p_B \circ f) \in A^X \times B^X. \tag{6}$$

This assignment is one-to-one and onto (it can be easily verified), hence

$$(A \times B)^X \sim A^X \times B^X \tag{7}$$

and

$$(ab)^x = a^x b^x. \tag{8}$$

● PROBLEM 6–29

Prove

$$2^{\text{card } N} = \text{card } R. \tag{1}$$

SOLUTION:

Recall that the symbols, card N and card R, denote the cardinal numbers of the sets N and R, respectively. Equation (1) will be proven in two steps: first we shall show that

$$2^{\text{card } N} \geq \text{card } R \tag{2}$$

and then we shall prove

$$\text{card } R \geq 2^{\text{card } N}. \tag{3}$$

Let Q represent the set of rational numbers; then

$$N \sim Q. \tag{4}$$

The function

$$f : R \to P(Q) \tag{5}$$

defined by

$$f(x) = \{y \in Q : y < x\} \tag{6}$$

is one-to-one. Indeed, if $x, x' \in R$ and $x < x'$, then a rational number $y \in Q$ exists, such that

$$x < y < x'. \tag{7}$$

The rational numbers are a dense subset of the set of real numbers.

Then $y \in f(x')$, but $y \notin f(x)$, thus, $f(x) \neq f(x')$ and f is one-to-one.

For any two sets, A and B, card $A \leq$ card B, if and only if an injection $f : A \to B$ exists. Hence,

$$\text{card } R \leq \text{card } P(Q) = 2^{\text{card } Q} = 2^{\text{card } N}. \tag{8}$$

To prove card $R \geq 2^{\text{card } N}$, define

$$F : \{0, 1\}^N \to R \tag{9}$$

such that

$$F(f) = 0.f(1)\,f(2)\,f(3)\,\dots \tag{10}$$

where $f : N \to \{0, 1\}$. $F(f)$ is a decimal number consisting of 0's and 1's. Function F is one-to-one. Suppose $f, g \in \{0, 1\}^N$ and $f \neq g$, then $F(f) \neq F(g)$ because decimals defining $F(f)$ and $F(g)$ are different. Therefore,

$$\text{card}\,\{0, 1\}^N \leq \text{card}\,R \tag{11}$$

or

$$2^{\text{card}\,N} \leq \text{card}\,R \tag{12}$$

Q.E.D.

● PROBLEM 6–30

Prove

$$\text{card}\,N < \text{card}\,R. \tag{1}$$

SOLUTION:

According to Cantor's theorem (If X is a set, then $\text{card}\,X < \text{card}\,P(X)$) we have

$$\text{card}\,N < \text{card}\,P(N). \tag{2}$$

For any set A, the following equality holds:

$$\text{card}\,P(A) = 2^{\text{card}\,A} \tag{3}$$

(see Problem 6-25). Thus

$$\text{card}\,N < \text{card}\,P(N) = 2^{\text{card}\,N} =$$

$$= \text{card}\,R. \tag{4}$$

● PROBLEM 6–31

Show that for any finite $n \geq 2$

$$n^{\text{card}\,N} = (\text{card}\,N)^{\text{card}\,N} = \text{card}\,R. \tag{1}$$

SOLUTION:

We proved (see Problem 6-29)

$$\text{card}\,R = 2^{\text{card}\,N}. \tag{2}$$

Since $n \geq 2$,

$$2^{\text{card } N} \leq n^{\text{card } N} \tag{3}$$

and

$$n^{\text{card } N} \leq (\text{card } N)^{\text{card } N}. \tag{4}$$

In Problem 6-30, we proved

$$\text{card } N < 2^{\text{card } N}. \tag{5}$$

Combining (2), (3), (4), and (5)

$$\text{card } R = 2^{\text{card } N} \leq n^{\text{card } N} \leq (\text{card } N)^{\text{card } N} \leq (2^{\text{card } N})^{\text{card } N} =$$

$$= 2^{\text{card } N \cdot \text{card } N} = \tag{6}$$

but card $N \cdot$ card N = card N (see Problem 6-19). Hence

$$2^{\text{card } N \cdot \text{card } N} = 2^{\text{card } N} = \text{card } R. \tag{7}$$

From (7), we conclude that

$$\text{card } R = n^{\text{card } N} = (\text{card } N)^{\text{card } N}. \tag{8}$$

● PROBLEM 6-32

Prove that:

1. card C = card R (1)
   where C is the set of complex numbers.

2. card $R \cdot$ card N = card R. (2)

SOLUTION:

1. A complex number

$$z = x + iy \tag{3}$$

can be represented by an ordered pair

$$(x, y)$$

where x and y are real numbers. Thus,

$$C \sim R \times R \tag{4}$$

and

$$\text{card } C = \text{card } R \cdot \text{card R}. \tag{5}$$

From Problem 6-21

$$\text{card } R \cdot \text{card } R = \text{card } R. \tag{6}$$

Hence

$$\text{card } C = \text{card } R. \tag{7}$$

2. It is easy to verify that

$$\text{card } R \le \text{card } R \cdot \text{card } N \le \text{card } R \cdot \text{card } R = \text{card } R. \tag{8}$$

Therefore

$$\text{card } R \cdot \text{card } N = \text{card } R. \tag{9}$$

● PROBLEM 6-33

In Problem 6-21, we proved

$$\text{card } R \cdot \text{card } R = \text{card } R. \tag{1}$$

Using

$$2^{\text{card } N} = \text{card } R \tag{2}$$

prove (1).

SOLUTION:

For three cardinal numbers a, b, and c

$$a^b a^c = a^{b+c} \tag{3}$$

(see Problem 6-26).

Thus, from (2) and (3), we obtain

$$\text{card } R \cdot \text{card } R = 2^{\text{card } N} 2^{\text{card } N} = 2^{\text{card } N + \text{card } N} =$$

$$= 2^{\text{card } N} = \text{card } R. \tag{4}$$

Note that in Problem 6-15 we proved

$$\text{card } N + \text{card } N = \text{card } N. \tag{5}$$

● PROBLEM 6-34

Let

$$F = \{f : f : R \to R\} \tag{1}$$

represent a set of all functions from R to R. Compare the cardinal numbers, card F and card R.

SOLUTION:

According to the definition of exponentiation of cardinal numbers, we have

$$\text{card}\, F = \text{card}\, R^R = (\text{card}\, R)^{\text{card}\, R} \tag{2}$$

(see Problem 6-23).

Since

$$2^{\text{card}\, N} = \text{card}\, R \tag{3}$$

(2) leads to

$$\text{card}\, F = (\text{card}\, R)^{\text{card}\, R} = (2^{\text{card}\, N})^{\text{card}\, R} =$$

$$= 2^{\text{card}\, N \cdot \text{card}\, R}. \tag{4}$$

From Problem 6-32, we obtain

$$\text{card}\, N \cdot \text{card}\, R = \text{card}\, R \tag{5}$$

therefore

$$\text{card}\, F = 2^{\text{card}\, R}. \tag{6}$$

According to Cantor's theorem (for any set X, card $X <$ card $P(X)$) and

$$\text{card}\, P(X) = 2^{\text{card}\, X} \tag{7}$$

we find setting $X = R$

$$2^{\text{card}\, R} = \text{card}\, P(R) > \text{card}\, R. \tag{8}$$

From (6) and (8), we obtain

$$\text{card}\, F = 2^{\text{card}\, R} > \text{card}\, R. \tag{9}$$

● PROBLEM 6-35

Prove that

$$\text{card}\, C(R, R) = \text{card}\, C(Q, R) =$$

$$= \text{card}\, K(R, R) = \text{card}\, R \tag{1}$$

where $C(R, R)$ and $C(Q, R)$ are the sets of continuous real-valued functions with domains R and Q, respectively. The set of all real-valued constant functions with the domain R is denoted by $K(R, R)$.

SOLUTION:

Note that the each function $f : R \to R$ a function corresponds $f \mid Q : Q \to R$, such that for all $x \in Q$, $f(x) = f \mid Q(x)$. Function $f \mid Q$ is the restriction of f to Q. We denote this correspondence by

$$F : C(R, R) \to C(Q, R). \tag{2}$$

The restriction of a continuous function is a continuous function. We show that F is one-to-one. Indeed, let $f, g \in C(R, R)$ denote such that for all $x \in Q$

$$f(x) = g(x). \tag{3}$$

Let x' be any real number, $x' \in R$. Then a sequence $(x_n) \in Q$ of rational numbers exists, such that

$$\lim_{n \to \infty} x_n = x' \tag{4}$$

Since both f and g are continuous

$$f(x') = g(x'). \tag{5}$$

Therefore, $f = g$. Function F is injective, thus

$$\text{card}\, C(R, R) \leq \text{card}\, C(Q, R). \tag{6}$$

On the other hand,

$$\text{card}\, C(Q, R) \leq \text{card}\, R^Q = (\text{card}\, R)^{\text{card}\, N} =$$

$$= (2^{\text{card}\, N})^{\text{card}\, N} = 2^{\text{card}\, N} = \text{card}\, R. \tag{7}$$

From (6) and (7),

$$\text{card}\, C(R, R) \leq \text{card}\, C(Q, R) \leq \text{card}\, R. \tag{8}$$

Consider the set $K(R, R)$. To each real number $p \in R$, a constant function corresponds

$$f_p : R \to R \tag{9}$$

such that for all $x \in R$, $f_p(x) = p$. Thus

$$\text{card}\, K(R, R) = \text{card}\, R. \tag{10}$$

Each constant function is continuous

$$K(R, R) \subset C(R, R). \tag{11}$$

Hence

$$\text{card}\, R = \text{card}\, K(R, R) \leq \text{card}\, C(R, R). \tag{12}$$

We conclude from (8) and (12) that

$$\text{card}\, R = \text{card}\, C(R, R) = \text{card}\, C(Q, R) = \text{card}\, K(R, R). \tag{13}$$

Surprisingly, there are as many constant functions as continuous functions.

● PROBLEM 6–36

Let $D(R, R)$ represent the set of all differentiable real-valued functions. Show that

$$\text{card}\, D(R, R) = \text{card}\, R. \tag{1}$$

SOLUTION:

Each differentiable function is continuous

$$D(R, R) \subset C(R, R). \tag{2}$$

Hence

$$\text{card}\, D(R, R) \leq \text{card}\, C(R, R) = \text{card}\, R \tag{3}$$

(see Problem 6-35).

Each constant function is differentiable

$$K(R, R) \subset D(R, R) \tag{4}$$

hence

$$\text{card}\, R = \text{card}\, K(R, R) \leq \text{card}\, D(R, R) \tag{5}$$

(see Problem 6-35). Thus

$$\text{card}\, D(R, R) = \text{card}\, R. \tag{6}$$

● PROBLEM 6–37

The so-called classic Hilbert space, H, consists of all infinite sequences

$$(x_1, x_2, x_3, \ldots)\ ;\ x_k \in R\ ;\ k \in N \tag{1}$$

such that the series

$$x_1^2 + x_2^2 + x_3^2 + \ldots \tag{2}$$

converges. Find card H.

SOLUTION:

Define function

$$f: R \to H \tag{3}$$

such that for every $x \in R$

$$f(x) = (x, 0, 0, 0, \ldots). \tag{4}$$

The infinite sequence belongs to the Hilbert space, H,

$$(x, 0, 0, \ldots) \in H. \tag{5}$$

Function f is an injection, hence

$$\text{card } R \leq \text{card } H. \tag{6}$$

On the other hand,

$$\text{card } H \leq (\text{card } R)^{\text{card } N} = (2^{\text{card } N})^{\text{card } N} =$$

$$= 2^{\text{card } N \cdot \text{card } N} = 2^{\text{card } N} = \text{card } R. \tag{7}$$

From (6) and (7), we conclude that

$$\text{card } H = \text{card } R. \tag{8}$$

● PROBLEM 6-38

Compare the cardinal numbers of these two sets:

1. The set of all infinite sequences

$$(x_1, x_2, x_3, \ldots) \tag{1}$$

of real numbers, denoted by $R^{\text{card } N}$.

2. The set of all infinite sequences of integers.

SOLUTION:

The cardinal number of the set of all infinite sequences of integers is

$$(\text{card } N)^{\text{card } N}. \tag{2}$$

From Problem 6-31, we obtain

$$(\text{card } N)^{\text{card } N} = \text{card } R. \tag{3}$$

The cardinal number of the set $R^{\text{card } N}$ is

$$\text{card } R^{\text{card } N} = (\text{card } R)^{\text{card } N} \tag{4}$$

Equation (4) leads to

$$(\text{card } R)^{\text{card } N} = (2^{\text{card } N})^{\text{card } N} = 2^{\text{card } N \cdot \text{ card } N} =$$

$$= 2^{\text{card } N} = \text{card } R. \tag{5}$$

Therefore, both sets have the same cardinal numbers. Sometimes, points

$$(x_1, x_2, x_3, \ldots)$$

where all x_k's are integers are called lattice points.

● PROBLEM 6–39

Compare the cardinal numbers of the following sets:

1. The set of all functions of one real variable, which assume values of 0 or 1.

2. The set of all real-valued functions of n real variables.

SOLUTION:

Consider the set of all functions

$$f(x) = \begin{cases} 0 \\ 1 \end{cases} \qquad f: R \to \{0, 1\}.$$

The cardinal number of this set is

$$\text{card } 2^R = 2^{\text{card } R}. \tag{1}$$

The cardinal number of the set of real-valued functions of n real variables

$$f: R^n \to R \tag{2}$$

is

$$\text{card } R^{R^n} = \text{card } R^{\text{card } R^n} \tag{3}$$

Since

$$(\text{card } R)^n = \text{card } R \tag{4}$$

we have

$$\text{card } R^{\text{card } R^n} = \text{card } R^{\text{card } R} = (2^{\text{card } N})^{\text{card } R} \tag{5}$$

but

$$\operatorname{card} N \operatorname{card} R = \operatorname{card} R \tag{6}$$

hence

$$\operatorname{card} R^{\operatorname{card} R^{n}} = 2^{\operatorname{card} R}. \tag{7}$$

Both sets have the same cardinal number.

● PROBLEM 6–40

Explain the continuum hypothesis and the generalized continuum hypothesis.

SOLUTION:

We recall briefly the major facts concerning cardinal numbers. There are finite cardinal numbers (which are non-negative integers) and transfinite cardinal numbers. The smallest transfinite cardinal number is card N (denoted $\aleph_0$, aleph-null). There is no largest cardinal number. Let us denote card $R = c$

$$\underbrace{0 < 1 < 2 < \ldots}_{\text{finite cardinal numbers}} \qquad \underbrace{< \aleph_0 < \ldots < c < \ldots < 2^c < \ldots}_{\text{transfinite cardinal numbers}}$$

We proved that $\aleph_0 < 2^{\aleph_0} = c$.

The following question posed by Cantor is known as the continuum problem: Is there a cardinal number that lies between $\aleph_0$ and c? Many mathematicians have spent many hours trying to solve this problem. Nobody has found a set A, such that card $A = a$ and $\aleph_0 < a < c$. Hence, the hypothesis is that such a set does not exist.

CONTINUUM HYPOTHESIS

There is no cardinal number p, such that

$$\aleph_0 < p < c = 2^{\aleph_0}.$$

A similar question exists for any transfinite cardinal number: Is there a cardinal number that lies between a transfinite cardinal number p and 2^p? This question has not been answered.

For any transfinite cardinal number p, no cardinal number r has been found, such that $p < r < 2^p$. Thus:

GENERALIZED CONTINUUM HYPOTHESIS

For any transfinite cardinal number p, there is no cardinal number r, such that

$$p < r < 2^p.$$

● PROBLEM 6–41

Do we know the solution of the continuum problem?

SOLUTION:

The continuum problem was formulated by Cantor in 1880. It became one of the most famous unresolved mathematical problems. On Hilbert's list of 23 mathematical questions with no answers, it occupied first place. (This list was announced in 1900 at the International Congress of Mathematicians in Paris.)

In 1938, Kurt Gödel proved that the generalized continuum hypothesis is consistent with the axioms of set theory. What this means can be explained as follows: Consider two systems of axioms

A – axioms for set theory, and

B – all axioms of A with the generalized continuum hypothesis added.

Any contradiction that might be implied by system B could be formulated as a contradiction implied by system A.

Later, P.J. Cohen proved that the negation of the continuum hypothesis is also consistent with the axioms for set theory. Thus, both the continuum and generalized continuum hypotheses are independent of the other axioms for set theory. They cannot be proved or disproved on the basis of these axioms.

W. Sierpinski has shown that, if the generalized continuum hypothesis is added as an axiom to the axioms of set theory, then the axiom of choice becomes redundant. It can be derived from the generalized continuum hypothesis and remaining axioms.

The situation is that the generalized continuum hypothesis is not provable on the basis of the axioms of set theory. We may postulate the generalized continuum hypothesis or deny it and, in either case, obtain a consistent mathematical theory.